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CONTENTS

On the Computation of Mathieu Functions 1
By Gertrude Blanch

Tensors and Equivalent Circuits 21
By Banesh Hoffman

Auxiliary Table of Complete Elliptic Integrals 26
By E. L. Kaplan

A Note on Bessel Functions of Purely Imaginary Argument 37
By Elliott W. Montroll

Laguerre Functions in the Mathematical Foundations of the Electromagnetic Theory of the Paraboloidal Reflector 49
By Edmund Pinney

Stresses and Small Displacements of Shallow Spherical Shells. I 80
By Eric Reissner

Note on Coefficients for Numerical Integration with Differences 86
By Herbert E. Salzer

Reducible Dynamical Systems 89
By T. Y. Thomas

Table of Bessel Functions and Derivatives $J_2, J_1', J_2', N_2, N_1', N_2'$ 93
By H. B. Dwight

A Three-Dimentional Analog of a Plane Kempe Linkage 96
By Michael Goldberg

Analytic Expressions in Photometry and Colorimetry 111
By Parry Moon and Domina Eberle Spencer

On the Transformation of the Equations of Dynamics 191
By T. Y. Thomas

Mathieu Functions and Their Classification 209
By W. McLachlan

Note on The Shear Stresses in a Bent Cantilever Beam of Rectangular Cross Section 241
By E. Reissner and G. B. Thomas

Coefficients for Facilitating the Use of the Gaussian Quadrature Formula 244
By Herbert E. Salzer

Similarity Laws of Hypersonic Flows 247
By Hsue-Shen Tsien

Table of the Struve Functions $L_\nu(x)$ and $H_\nu(x)$ 252
By the Mathematical Tables Project

The Wiener (Root Mean Square) Error Criterion in Filter Design and Prediction 261
By Norman Levinson

Stresses and Small Displacements of Shallow Spherical Shells II 279
By Eric Reissner

One-Dimensional Flows of a Gas Characterized by Van der Waal's Equation of State 301
By Hsue-Shen Tsien

ON THE COMPUTATION OF MATHIEU FUNCTIONS

By Gertrude Blanch

Introduction

Ince, Goldstein, and others[1] have developed a method for obtaining the characteristic values of Mathieu's differential equation from a continued fraction. Schemes for generating the corresponding Fourier coefficients for the periodic solutions have also been indicated. But there does not seem to appear in the literature any method for improving the accuracy of the characteristic values, except by cumbersome iteration. In this paper, a method is outlined for correcting the characteristic values and the Fourier coefficients, in the process of generating the latter. The expression for the error in the ratio of two successive coefficients, in terms of the error in the characteristic value, is also given. Section I is devoted to a detailed description of the method to be used in generating the Fourier coefficients on the basis of an approximate characteristic value. In Section II, expresions for the error in the characteristic value and in the ratios of the Fourier coefficients are obtained. In Section III, a systematic method for obtaining the corrected Fourier coefficients is developed, based on the results obtained in Section II. Two illustrative examples are provided.

Although this paper deals entirely with Mathieu functions, the method is applicable in other types of equations, when the coefficients of the power series solution or of the Fourier series are determined by a three-term recursion formula.

In this discussion it will be convenient to write Mathieu's differential equation in the form

$$\frac{d^2y}{dt^2} + (\alpha - 2\theta \cos 2t)y = 0. \tag{1}$$

It is known that when α belongs to a certain countable set of characteristic values, equation (1) is satisfied by one of the following periodic solutions:

$$y = \sum_{n=0}^{\infty} B_{2n} \cos 2nt \qquad \text{(Even solutions of period } \pi) \tag{2a}$$

$$y = \sum_{n=0}^{\infty} B_{2n+1} \cos (2n+1)t \qquad \text{(Even solutions of period } 2\pi) \tag{2b}$$

$$y = \sum_{n=1}^{\infty} B_{2n} \sin 2nt \qquad \text{(Odd solutions of period } \pi) \tag{2c}$$

$$y = \sum_{n=0}^{\infty} B_{2n+1} \sin (2n+1)t \qquad \text{(Odd solutions of period } 2\pi). \tag{2d}$$

The same symbol B has been used for the coefficients in (2*a*)–(2*d*), for convenience in discussing operations which apply to all four solutions. A charac-

[1] See "Tabulation of Mathieu Functions" by W. G. Bickley in *Mathematical Tables and Other Aids to Computation* July, 1945.

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teristic value α gives rise to one and only one periodic solution, when θ is different from zero, and the values of B in (2*a*), say, are not related in any way to the corresponding values of B in (2*c*).

I. Method of Generating the Coefficients

Assuming that a value of α has been calculated (to a certain number of decimal places) which determines one of these four types of solutions (2a)–(2d), we shall outline a method for computing the corresponding Fourier coefficients, and improving the accuracy of the characteristic value if necessary.
Let

$$G_m = B_m/B_{m-2}\,; \qquad H_m = 1/G_m\,; \tag{3}$$

$$k_m = (\alpha - m^2)/\theta. \tag{4}$$

From the above definitions and (1), the following relations may be readily verified:

$$G_m = k_{m-2} - H_{m-2} \qquad m \geq 5 \tag{5a}$$

$$G_m = 1/(k_m - G_{m+2}) \qquad m \geq 3 \tag{5b}$$

$$\left.\begin{aligned} G_2 &= k_0\,, \\ G_4 &= k_2 - 2H_2 = 1/(k_4 - G_6) \end{aligned}\right\} \quad \text{for solutions of type (2a)} \tag{6a}$$

$$G_3 = (k_1 - 1) = 1/(k_3 - G_5), \quad \text{for solutions of type (2b)} \tag{6b}$$

$$\left.\begin{aligned} H_2 &= 0; \\ G_4 &= k_2 = 1/(k_4 - G_6), \end{aligned}\right\} \quad \text{for solutions of type (2c)} \tag{6c}$$

$$G_3 = (k_1 + 1) = 1/(k_3 - G_5), \quad \text{for solutions of type (2d)} \tag{6d}$$

It should be observed that since the series (2a)–(2d) converge absolutely for finite values of θ, there exists an integer m_0 for each series such that

$$|G_m| < 1, \qquad m \geq m_0\,. \tag{7}$$

In order to fix ideas, we shall confine our attention for the present to solutions of type (2a).

Apart from an arbitrary factor, the coefficients B_m of (2a) are determined by (5) and (6a). It might therefore seem reasonable to compute in turn G_2, G_4, $\cdots$, until all required values of G_m are available, and then to compute the coefficients AB_0, AB_2, AB_4, $\cdots$, where A is an arbitrary constant. This process, however, is unsatisfactory. For let us attempt to compute G_m for a value of $m \geq m_0$ of (7), where k_m is numerically large.

Since for such values of m and k_m

$$|G_m| = |k_{m-2} - H_{m-2}| < 1,$$

it follows that H_{m-2} must be of the same sign and order of magnitude as k_{m-2}, and there will therefore be a loss in significant figures in the subtraction of H_{m-2} from k_{m-2}. It is true that the values of B_m also decrease in magnitude in this region, but the decrease is not as rapid as the loss of accuracy in G_m, if the latter is computed from (5a). The reader may verify this from the last example given at the end of this paper. We shall therefore modify the process of computing the successive values of G_m to overcome this difficulty.

For the sake of clarity, consider for the moment the case where the coefficients B_m begin with positive values and increase up to the largest coefficient B_r, and that thereafter

$$B_{r+n} < B_{r+n-2}, \qquad n \geq 2.$$

This implies that

$$G_{r+n} < 1, \qquad G_{r-n} > 1, \qquad n > 0.$$

Let us further assume that for the desired accuracy, it is possible to terminate the series (2a) with the coefficient B_w; the number w which will suffice may be determined quite readily after a little experience. The computations may then proceed in the following stages:

Stage 1. The values of $k_0, k_2, \cdots, k_w$, defined by equation (4), are computed to the number of significant figures required in the corresponding values of G_m. If it is not known apriori how many significant figures are needed in G_m, the values of k_m may be computed to the number of significant figures required in the largest coefficient B_m. To check the computations, the values of k_m may be differenced as a function of m.

Stage 2. The values $G_2, H_2, G_4, H_4, \cdots, G_r, H_r$ are computed from (6a) and (5a). Usually it will not be known apriori which coefficient B_r in the sequence is the largest; the computer therefore proceeds with the computation of the G's so long as they remain larger than unity; eventually he will reach a stage where G_r is the last one which is greater than unity. The process stops at this point after the determination of H_r. It should be pointed out that in this stage every H_m is obtained as the reciprocal of G_m. This method of generating successive values of G_m from preceding values may be termed the "forward" generating scheme.

Stage 3. If B_w is the last required coefficient in the series, then B_w will be a number containing few significant figures. In this region, k_w will be numerically large, so that to at least three significant figures, we shall have[2]

$$G_w \cong 1/k_w.$$

[2] The expression for G_w as a continued fraction is

$$G_w = \frac{1}{k_w - \dfrac{1}{k_{w+2} - \dfrac{1}{k_{w+4} - \cdots}}}.$$

After the value of G_w has been determined, the following may be computed in turn:

$$(8)\qquad \begin{cases} H_{w-2} = k_{w-2} - G_w\,;\ G_{w-2} = 1/H_{w-2}\,;\ H_{w-4} = k_{w-4} - G_{w-2}\,; \\ G_{w-4} = 1/H_{w-4}\,; \\ \qquad\qquad \cdots \quad \cdots \\ H_r = k_r - G_{r+2}\,, \qquad r \geq 4;\ \ H_2 = \tfrac{1}{2}(k_2 - G_4), \text{ if } r = 2. \end{cases}$$

This process may be termed the "backward" generating scheme, and this time G_m is obtained as the reciprocal of H_m. In this stage of the calculations, an increased number of significant figures is gained in the first few steps until the maximum required accuracy is reached. Thus if $k_{w-2} = 30$, and $G_w \cong .03$, $H_{w-2} = 29.27$ (correct to four significant figures); hence G_{w-2} will be approximately correct to five decimals, and H_{w-4} will be correct to the same number of decimals, or to seven significant figures, etc. (assuming that k_w is known that accurately).

The last value obtained in this process is H_r, which is compared with the value obtained in stage 2. If the agreement is satisfactory, the computer may go on to the next stage of determining the coefficients themselves.[3]

The agreement of the two values of H_r constitutes a powerful check on the accuracy of the characteristic value, and serves as a partial check on the accuracy of the values of G_m and H_m. Nevertheless, it is possible for the two values of H_r to agree to the required accuracy, in spite of an error in some value of H_{r+m}, if the error is small enough not to affect the accuracy of H_r itself. For this reason, it is desirable to check carefully the computations of G_m and H_m. The operations are in fact very simple, since they consist of a subtraction followed by a division. Each subtraction and division should be performed twice at every step of the process, especially if high accuracy is required, and the divisions are made with more than ten significant figures.

We have assumed that the ratios G_m are positive and greater than one, up to a certain point, and then steadily decrease. This is not always so. In some cases, the ratios G_m may start with small values, and keep on decreasing numerically. In that case B_0 will be the largest coefficient, and stage 2 stops with H_2. In other cases, G_2 may be positive or negative, and some value G_p may become very large numerically (in the neighborhood where B_{p-2} approaches zero). In this case, one may end stage 2 with H_p, or an earlier value of H_m. An example of this type is given at the end of the present paper.

Stage 4. After the values of G_m and H_m are available, we determine a set of coefficients B_0, B_2, $\cdots$, B_w which may subsequently be normalized in any

[3] Goldstein ("Mathieu Functions" *Cambridge Phil. Soc. Trans.* Vol. XXIII, pp. 303-336) uses a scheme similar to this one. In computing the Fourier coefficients corresponding to a characteristic value of the r^{th} order, he recommends generating the successive ratios, starting with a ratio v_r, which in his paper is analogous to G_r. It happens that for θ not too large, B_r is one of the largest numerical coefficients in the r^{th} characteristic function.

required manner. If H_r is the ratio computed by both the forward and backward processes, let

(9a) $$B_r = 1.$$

All other coefficients are then completely determined. Thus

(9b) $$B_{r+2} = G_{r+2}, \quad B_{r+4} = B_{r+2}G_{r+4}, \quad \cdots, \quad B_w = B_{w-2}G_w.$$

(9c) $$B_{r-2} = H_r, \quad B_{r-4} = B_{r-2}H_{r-2}, \quad \cdots, \quad B_0 = B_2H_2, \quad r \geq 2.$$

If $r = 0$,

$$B_0 = 1, \qquad B_2 = G_2, \text{ etc.}$$

The computations may be checked by applying the recurrence relations

(10a) $$k_m B_m = B_{m-2} + B_{m+2}, \qquad m \geq 4$$

(10b) $$k_2 B_2 = 2B_0 + B_4.$$

When solutions of type (2b), (2c), or (2d) are considered, equation (6a) is replaced by (6b), (6c), or (6d) for purposes of computing the successive ratios G_m and H_m in stages 2 and 3. $H_2 = 0$ for solutions of type (2c), and G_3 is the first ratio obtained in solutions of type (2b) and (2d). Equation (10b) is replaced by

$$k_3 B_3 = B_1 + B_5 \text{ (for solutions of type 2b and 2d).}$$

Stage 5. The final step is to normalize the coefficients. For example, let it be required to compute the coefficients of

$$Se_r(s, \cos t) = \sum_{m=0,1}^{\infty}{}' De_m^r \cos mt; \qquad \sum_{m=0,1}^{\infty}{}' De_m^r = 1$$

where $s = 4\theta$, and $\sum'$ implies summation over even values of m, if r is even, and over odd values of m if r is odd.[4] To obtain these coefficients, we compute

$$A = 1 \Big/ \left(\sum_{m=0,1}^{\infty}{}' B_m \right).$$

Clearly $De_m^r = AB_m$. The coefficients De_m^r may be checked by applying formulas (10a) and (10b) with B replaced by De^r, and also by differencing the values of De_m^r as a function of s. Similarly, for computing the function

$$So_r(s, \cos t) = \sum_{m=1,2}^{\infty}{}' Do_m^r \sin mt;$$

where

$$\sum_{m=1,2}^{\infty}{}' mDo_m^r = 1,$$

[4] Normalization used by Stratton-Morse-Chu-Hutner.

let

$$A = 1 \Big/ \Big(\sum_{m=1,2}^{\infty}{}' mB_m \Big);$$

it follows that $Do_m^r = 4B_m$, for every m.

For Ince's normalization, let

$$A = \pm (2B_0^2 + B_2^2 + \cdots)^{-\frac{1}{2}} \text{ or } \pm (B_1^2 + B_2^2 + \cdots)^{-\frac{1}{2}},$$

the sign to be so determined that $ce_r(x, \theta) > 0$, $\frac{d}{dx} se_r(x, \theta) > 0$, when $x = 0$.

II. Determination of the Error in the Characteristic Value and in the Ratios G_m and H_m

The error in the ratios G_m is due to two main causes:

(a) The cumulative rounding error due to the various computing operations.

(b) The error in α.

The error due to (a) may be compensated for by keeping a sufficient number of significant figures in the computations. The effect on G_m of the initial error in α will now be determined; and for this purpose we shall assume that enough significant figures are kept in the subsidiary computations, so that the error in the last decimal place of interest is due only to the error in α.

Let

$$\alpha = \bar{\alpha} + \lambda, \tag{11}$$

where α is the true value, and $\bar{\alpha}$ the computed value, which is in error by λ. As before, the analysis will be carried forward for the case of the coefficients of (2a); the variations in the formulas corresponding to the other three types of solutions will be given at the end of the discussion.

Let $\epsilon_{m,1}$ and $\eta_{m,1}$ be the errors in G_m and H_m, respectively, where G_m and H_m are the ratios generated in stage 2, by the forward process. Let the computed values be denoted by $G_{m,1}$ and $H_{m,1}$, corresponding to the true values G_m and H_m, respectively. The phrase "true value" shall be construed as a value of G_m or H_m based on the *exact* value of α; the value of G_m or H_m may still contain errors due to roundings or other causes which are not considered here.

Thus

$$\epsilon_{m,1} = G_m - G_{m,1}; \qquad \eta_{m,1} = H_m - H_{m,1}. \tag{12}$$

Since $G_2 = \alpha/\theta$, we at once have

$$\epsilon_{2,1} = \lambda/\theta;$$

Again since

$$H_2 = 1/(G_{2,1} + \epsilon_{2,1}),$$

it follows that

$$\eta_{2,1} \cong \epsilon_{2,1} H_{2,1}^2 \cong -\frac{\lambda}{\theta} H_2^2 .$$

In deriving $\eta_{2,1}$, we have neglected higher powers of $\frac{\lambda}{\theta}$ and $H_{2,1}$ has been replaced by H_2, for the sake of simplicity in notation. For purposes of computing $\eta_{2,1}$, the value of $H_{2,1}$ is generally a sufficiently exact approximation to H_2 when the terms involving $(\lambda/\theta)^2$ are neglected. In the succeeding analysis, we shall also write an equality sign in the equations for $\epsilon_{m,1}$ and $\eta_{m,1}$ in place of the more exact symbol of approximate equality.

By considering the method used to compute the ratios G_m and H_m in the forward process of stage 2, we may at once verify the following expressions for the successive values of $\epsilon_{m,1}$ and $\eta_{m,1}$:

$$(13)\quad \begin{cases} \epsilon_{2,1} = \lambda/\theta; \qquad \eta_{2,1} = -\dfrac{\lambda}{\theta} H_2^2 \\ \epsilon_{4,1} = \dfrac{\lambda}{\theta}(1 + 2H_2^2); \qquad \eta_{4,1} = -\dfrac{\lambda}{\theta}(H_4^2 + 2H_4^2 H_2^2) \\ \qquad\qquad \cdots\cdots \qquad\qquad \cdots\cdots \\ \epsilon_{m,1} = \dfrac{\lambda}{\theta}(1 + H_{m-2}^2 + H_{m-2}^2 H_{m-4}^2 + H_{m-2}^2 H_{m-4}^2 H_{m-6}^2 + \cdots \\ \qquad\qquad + 2H_{m-2}^2 \cdots H_2^2), \; m \geq 6 \\ \eta_{m,1} = -H_m^2 \epsilon_{m,1} . \end{cases}$$

Let

$$(14)\quad \begin{cases} R_{m,1} = H_m^2 + H_m^2 H_{m-2}^2 + H_m^2 H_{m-2}^2 H_{m-4}^2 + \cdots + 2H_m^2 \cdots H_2^2, \; m \geq 4 \\ R_{2,1} = H_2^2, \; R_{0,1} = 0. \end{cases}$$

From the above definition, it follows that

$$(15)\quad \begin{cases} \epsilon_{m,1} = \dfrac{\lambda}{\theta}(1 + R_{m-2,1}), \qquad m \geq 6 \\ \epsilon_{4,1} = \dfrac{\lambda}{\theta}(1 + 2R_{2,1}); \qquad \epsilon_{2,1} = \lambda/\theta \\ \eta_{m,1} = -\dfrac{\lambda}{\theta} R_{m,1} . \end{cases}$$

The following recurrence relations hold:

$$(16a)\qquad R_{m,1} = H_m^2(1 + R_{m-2,1}), \qquad m \geq 6$$

$$(16b)\qquad R_{4,1} = H_4^2(1 + 2R_{2,1}).$$

Let us now turn to the errors in the values of G_m and H_m which are derived in stage 3. This time the computing process starts with the last ratio of the set, called G_w, from the relation

$$G_w \cong \frac{1}{k_w}.$$

Let the values of H_m and G_m computed in stage 3 be termed $H_{m,2}$ and $G_{m,2}$, respectively, and let us define quantities $\epsilon_{m,2}$ and $\eta_{m,2}$ by the relations

$$\eta_{m,2} = H_m - H_{m,2}\,; \qquad \epsilon_{m,2} = G_m - G_{m,2}\,, \tag{17}$$

where H_m and G_m are the true values of the ratios in question. Since only a few decimal places are kept in the last ratio G_w, the error $\epsilon_{w,2}$ will be far smaller than the rounding-off error; but we shall determine $\epsilon_{w,2}$ just the same, even though the rounding-off error will be completely neglected, in keeping with the aim to consider here only the effect of an error in α. With this understanding, we may establish the results below, as a consequence of the method by which H_m and G_m were generated in stage 3.

$$\begin{cases} \epsilon_{w,2} = -\dfrac{\lambda}{\theta} G_w^2 \\ \eta_{w-2,2} = \dfrac{\lambda}{\theta}(1 + G_w^2); \qquad \epsilon_{w-2,2} = -\dfrac{\lambda}{\theta}(G_{w-2}^2 + G_{w-2}^2 G_w^2) \\ \eta_{m,2} = \dfrac{\lambda}{\theta}(1 + G_{m+2}^2 + G_{m+2}^2 G_{m+4}^2 + \cdots + G_{m+2}^2 \cdots G_w^2), \qquad m \geq 4 \\ \eta_{2,2} = \dfrac{\lambda}{\theta}\tfrac{1}{2}(1 + G_4^2 + G_4^2 G_6^2 + \cdots + G_4^2 G_6^2 \cdots G_w^2) \\ \epsilon_{m,2} = -G_m^2 \eta_{m,2}\,. \end{cases} \tag{18}$$

From the law of formation of the above expressions, it is apparent that if an infinite number of ratios G_m were considered, the coefficients of (λ/θ) in $\eta_{m,2}$ would involve the p-th term $G_{m+2}^2 \cdot G_{m+4}^2 \cdots G_{m+p}^2$, where the number of products tend to infinity with p. Hereafter this generalization will be introduced, since it simplifies the formal expression for $\eta_{m,2}$. However, no additional accuracy is gained by this device, because terms involving $(\lambda/\theta)^2$ and higher powers of (λ/θ) are neglected. A similar generalization will hereafter be introduced in the expression for $\epsilon_{m,2}$.

Let

$$\begin{cases} R_{m,2} = 1 + G_{m+2}^2 + G_{m+2}^2 G_{m+4}^2 + \cdots + G_{m+2}^2 G_{m+4}^2 \cdots G_{m+p}^2 + \cdots) \; m \geq 4 \\ R_{2,2} = \tfrac{1}{2}(1 + G_4^2 + G_4^2 G_6^2 + G_4^2 G_6^2 G_8^2 + \cdots) \\ R_{0,2} = 2 + G_2^2 + G_2^2 G_4^2 + \cdots \end{cases} \tag{19}$$

From the above definitions and (18), we have

$$\eta_{m,2} = \frac{\lambda}{\theta} R_{m,2}, \qquad m \geq 2; \qquad \epsilon_{m,2} = -\frac{\lambda}{\theta}(R_{m-2,2} - 1), \quad m \geq 6$$

(20)

$$\epsilon_{4,2} = -\frac{\lambda}{\theta}(2R_{2,2} - 1).$$

The following simple recurrence relations hold among successive values of $R_{m,2}$:

(21a) $$R_{m,2} = 1 + G^2_{m+2} R_{m+2,2}, \qquad m \geq 4$$

(21b) $$R_{2,2} = \tfrac{1}{2}(1 + G_4^2 R_{4,2}); \; R_{0,2} = 2 + 2R_{2,2} \cdot G_2^2.$$

From (15) and (20), we have the following two equations from which to determine H_ν (the ratio obtained by both the forward and backward processes) as well as λ/θ.

$$H_\nu = H_{\nu,1} - \frac{\lambda}{\theta} R_{\nu,1}$$

$$H_\nu = H_{\nu,2} + \frac{\lambda}{\theta} R_{\nu,2}$$

Hence

(22) $$\frac{\lambda}{\theta} = (H_{\nu,1} - H_{\nu,2})/(R_{\nu,1} + R_{\nu,2})$$

(23) $$H_\nu = (R_{\nu,2} H_{\nu,1} + R_{\nu,1} H_{\nu,2})/(R_{\nu,1} + R_{\nu,2}).$$

If the ratios G_m and H_m are such that

$$G_m > 1, m \leq \nu; \qquad G_m < 1, \qquad m > \nu,$$

then from (14) and (19), it is apparent that $R_{\nu,1}$ will usually be less than unity, while $R_{\nu,2}$ will always be greater than unity, if ν is greater than 2. When $R_{\nu,1}$ is smaller than $R_{\nu,2}$, $H_{\nu,1}$ is obviously a closer approximation to H_ν then $H_{\nu,2}$.

So far the expressions for $R_{\nu,1}$ and $R_{\nu,2}$ have been determined for solutions of type (2a). It may be verified that all except $R_{2,2}$ also hold for solutions of type 2c; there will be no $\epsilon_{2,1}$ or $\eta_{2,1}$, since G_4 and H_4 are the first required ratios. As for solutions of type (2b) and (2d), equations (16a) and (21a) hold for $m \geq 3$, if we set $R_{1,1} = 0$; this time $\epsilon_{3,1} = \frac{\lambda}{\theta}$, $\eta_{3,1} = -\frac{\lambda}{\theta} H_3^2$.

The expressions for $R_{m,1}$ of equation (14) and $R_{m,2}$ of equations (21a) may be written in terms of the coefficients B_m which were generated in stage 4. For, substituting the expression B_m/B_{m-2} for G_m, we have

(24a) $$R_{m,1} = (B^2_{m-2} + B^2_{m-4} + \cdots + 2B_0^2)/B_m^2, \qquad m \geq 4$$

(24b) $$R_{3,1} = B_1^2/B_3^2;$$

(24c) $$R_{2,1} = B_0^2/B_2^2$$

$$(25a)\qquad R_{m,2} = (B_m^2 + B_{m+2}^2 + \cdots)/B_m^2, \qquad m \geq 3$$

$$\left.\begin{aligned}(25b)\qquad R_{2,2} &= \tfrac{1}{2}(B_2^2 + B_4^2 + \cdots)/B_2^2\\ (25c)\qquad R_{0,2} &= 2B_0^2 + B_2^2 + \cdots)/B_0^2\end{aligned}\right\}\ \text{for even solutions of period } \pi$$

$$(25d)\qquad R_{2,2} = (B_2^2 + B_4^2 + \cdots)/B_2^2, \text{ for odd solutions of period } \pi$$

In particular, if $B_\nu = 1$, then for solutions of period π

$$(26a)\qquad R_{\nu,1} = B_{\nu-2}^2 + B_{\nu-4}^2 + \cdots + 2B_0^2, \qquad \nu \geq 4$$

For solutions of period 2π:

$$(26b)\qquad R_{\nu,1} = B_{\nu-2}^2 + B_{\nu-4}^2 + \cdots + B_1^2, \qquad \nu \geq 3$$

$$R_{\nu,2} = B_\nu^2 + B_{\nu+2}^2 + \cdots$$

Hence

$$(27)\qquad \begin{matrix}2B_0^2 + B_2^2 + B_4^2 + \cdots = \\ \text{or} \\ B_1^2 + B_3^2 + \cdots\end{matrix} \begin{cases} R_{\nu,1} + R_{\nu,2}, & \nu \neq 2 \\ 2(R_{2,1} + R_{2,2}) & \text{for even solutions of period } \pi, \text{ if } \nu = 2 \\ R_{\nu,1} + R_{\nu,2} & \text{for odd solutions of period } \pi \end{cases}$$

The quantities $R_{\nu,1}$ and $R_{\nu,2}$ are therefore useful for effecting Ince's normalization, or for obtaining the constant N_l of the Stratton-Morse-Chu-Hutner text.[5]

III. Method of Correcting the Coefficients

A systematic method of improving the accuracy of the final coefficients will now be outlined.

(a) The work outlined in Section I is carried through, except that the values of B_m computed in Stage 4 need not be checked by formulas (10) at this stage. It is sufficient for present purposes to compute the values of B_m to ten or fewer significant figures; but the ratio H_ν which is obtained in both stage 2 and stage 3 must be computed to one or two more significant figures than the number which will ultimately be required in $B_{\nu-2}$; some of the other ratios G_m and H_m may be computed to a lesser accuracy, depending on how rapidly the corresponding coefficients B_m fall off in magnitude.

(b) The successive values of $R_{m,1}$ and $R_{m,2}$ are computed with the aid of formulas (16) and (21). In addition, the value of f is obtained, defined by

$$(28)\qquad \begin{cases} f = 2B_0^2 + B_2^2 + \cdots & \text{(for solutions of period } \pi) \\ f = B_1^2 + B_3^2 + \cdots & \text{(for solutions of period } 2\pi) \end{cases}$$

As a check on the computations of $R_{m,1}$ and $R_{m,2}$, we have from (27) and (28)

$$R_{\nu,1} + R_{\nu,2} = f \qquad (\nu \neq 2 \text{ when even solutions are involved})$$

$$2(R_{2,1} + R_{2,2}) = f \qquad \text{in even solutions.}$$

[5] J. A. Stratton, P. M. Morse, L. J. Chu, and R. A. Hutner: "Elliptic Cylinder and Spheroidal Wave Functions" (1941) page 54.

In these operations, it is never necessary to carry more than ten significant figures.

(c) The corrected value of H_ν is obtained from (23) and λ is determined from (22), together with the corrected value of $\alpha = \bar{\alpha} + \lambda$ ($\bar{\alpha}$ is the approximate characteristic value with which the process started).

(d) The values of $k_m = (\alpha - m^2)/\theta$ are obtained on the basis of the corrected values of α, and differenced.

(e) The values of $G_{\nu+n}$ and $H_{\nu-n}$, $n > 0$, are corrected according to the following formulas:

$$(29)\quad \begin{cases} G_m = G_{m,2} - \dfrac{\lambda}{\theta}(R_{m-2,2} - 1), & m = \nu + 2, \nu + 4, \cdots; \quad m \geq 5 \\ G_4 = G_{4,2} - \dfrac{\lambda}{\theta}(2R_{2,2} - 1), & \text{for even solutions of period } \pi \\ G_4 = k_2, & \text{for odd solutions of period } \pi \\ G_2 = k_1 - 1, & \text{for even solutions of period } 2\pi \\ G_2 = k_1 + 1, & \text{for odd solutions of period } 2\pi \end{cases}$$

$$(30)\quad H_m = H_{m,1} - \frac{\lambda}{\theta} R_{m,1}.$$

It is not necessary to correct both G_m and H_m for the same m, as only one of the two ratios is required for generating the corrected values of B_m. If two calculating machines are employed in correcting the values of G_m and H_m, the value of λ/θ may be kept on the keyboard of one machine as a constant multiplier, the additions or subtractions involved in the corrections may be performed on the second machine, and only the final answer need be recorded.

As an added precaution against an error in the value of $\frac{\lambda}{\theta}$ or H_ν, where H_ν is the value corrected on the basis of formula (23), one may compute H_ν from the following two formulas

$$H_\nu = k_\nu - G_{\nu+2} = \frac{1}{k_{\nu-2} - H_{\nu-2}}, \qquad \nu \geq 5$$

$$\text{If } \nu = 4, \qquad H_4 = k_4 - G_6 = 1/(k_2 - 2H_2)$$

$$\text{If } \nu = 3 \text{ or } 1, \qquad H_3 = k_3 - G_5 = 1/(k_1 \mp 1),$$

the upper or lower sign holding, depending on whether the solution is even or odd. If $\nu = 2$ or 0: $H_2 = \dfrac{\theta}{\alpha} = \frac{1}{2}(k_2 - G_4)$, for even solutions of period π. The above values of H_ν, computed by the two formulas, should agree with (23) to the required accuracy. It is best to check the value of H_ν by the above method before correcting all required values of G_m and H_m; this will save unnecessary labor in case the values of H_ν fail to agree because of an error in the

value of $\frac{\lambda}{\theta^2}$, or in some value of G_m or H_m previously generated. The fact that agreement is obtained in the values of H_r does not necessarily mean that the value of H_r is correct. However, any error in the values of H_m will be discovered in the process of checking the values of B_m in the following stage.

(f) The values of B_m are then obtained on the basis of the corrected values of G_m and H_m, to the required number of significant figures; some of the values of B_m may need no corrections. After a value of B_m is computed, it should be checked by the relation

$$k_{m-2}B_{m-2} = B_{m-4} + B_m, \qquad m \geq 5$$

$$k_2 B_2 = B_4 + 2B_0.$$

If found correct, the succeeding value of B_m is computed and in turn tested. Finally, each value of B_m may be differenced as a function of θ (or $s = 4\theta$, if s is the argument in the table). The corrected values of B_m may then be normalized in any required manner.

Remarks

If the coefficients are required to more than ten significant figures, the above method of starting with an approximate value and doubling its accuracy (or otherwise improving it) is simpler than the process of generating sufficiently accurate characteristic values from the continued fraction. In the case where only one or two additional places are required, the suggested method of correction is especially simple. For the coefficients fall off in magnitude fairly rapidly; and it is usually enough to correct one or two of the ratios G_m and H_m which enter into the computation of the dominant coefficients B_m. Often an intelligent computer may make the corrections at the cost of only a few minutes' additional time.

In developing the method outlined here, the writer received much assistance from several members of the Mathematical Tables Project, especially from Miss Irene Stegun and Mrs. Beatrice Mittelman. The writer is also grateful to Miss Leona Freeman, Mrs. Dorothy Stein, and Mrs. Alice Adler, who cooperated in applying the method.

Examples

Two examples will be given: the first one will carry through the process by which nineteen decimal places may be obtained in the coefficients B_m; the second will illustrate how the process may be used to improve the accuracy of the coefficients in the case where one of the ratios G_m becomes very large numerically.

Example 1. Let it be required to compute the Fourier coefficients corresponding to the characteristic value of order zero, for $\theta = 24$, starting with $\alpha = -38.45897\ 3169$.

Stage 1. Values of k_m based on $\tilde{\alpha} = -38.45897\ 3169$	**Correction Process.** Values of k_m based on corrected value of $\alpha = -38.45897\ 31689\ 81452\ 4539$ (see "Correction Process" following stage 4)
k_0 : $-1.60245\ 72153\ 75000\ 0000$	$-k_0 = 1.60245\ 72153\ 74227\ 18558$
k_2 : $-1.76912\ 38820\ 41666\ 6667$	$-k_2 = 1.76912\ 38820\ 40893\ 85225$
k_4 : $-2.26912\ 38820\ 41666\ 6667$	$-k_4 =$ 20 ...
k_6 : $-3.10245\ 72153\ 75000\ 0000$	53 ...
k_8 : $-4.26912\ 388\cdot\cdot$	20 ...
k_{10}: $-5.76912\ 388\cdot\cdot\cdot$	20 ...
k_{12}: $-7.60245\ 72153\ 75$	53
k_{14}: $-9.76912\ 388\cdot\cdot\cdot$	20
k_{16}: $-12.26912\ 388\cdot\cdot\cdot$	20
k_{18}: $-15.10245\ 72153\ 75$	53
k_{20}: $-18.26912\ 388\cdot\cdot$	
k_{22}: $-21.76912\ 388\cdot\cdot$	
k_{24}: $-25.60245\ 722$	
k_{26}: $-29.76912\ 388$	
k_{28}: $-34.26912\ 388$	
k_{30}: $-39.10245\ 721$	
k_{32}: -44.26912	
k_{34}: -49.769	
$k_m = (\tilde{\alpha} - m^2)/\theta$	$k_m = (\alpha - m^2)/\theta$
The omitted digits in k_8, etc. are the same as in k_4.	The first 8 decimals are the same as in the uncorrected values at the left; the omitted digits at the end of the numbers are the same as in k_0 or k_2.

The second difference of k_m, as a function of m, should be constant.

Example 1

Stage 2 and Stage 3

Computation of G_m and H_m

Stage 2

$G_{2,1} = -1.60245\ 72153\ 75 = \dfrac{\tilde{\alpha}}{\theta}$; $H_{2,1} = -0.62404\ 16220\ 82237\ 24213 = 1/G_{2,1}$

Stage 3

$H_{2,2} = -0.62404\ 16220\ 83043\ 14185$

$G_{4,2} = -0.52104\ 0637\ 8755803830$

$G_{6,2} = -0.34988\ 7785\ 4537636311$

$G_{8,2}$ =	−0.24439	8027	332420334
$G_{10,2}$ =	−0.17743	7828	291975979
$G_{12,2}$ =	−0.13334	7060	232911531
$G_{14,2}$ =	−0.10322	9273	99149041
$G_{16,2}$ =	−0.08194	9291	60103403
$G_{18,2}$ =	−0.06645	5860	59678
$G_{20,2}$ =	−0.05487	5391	5910
$G_{22,2}$ =	−0.04601	9298	964
$G_{24,2}$ =	−0.03911	011598	
$G_{26,2}$ =	−0.03362	483682	
$G_{28,2}$ =	−0.02920	25924	
$G_{30,2}$ =	−0.02558	86294	
$G_{32,2}$ =	−0.0225994		
$G_{34,2}$ =	−0.0201		

NOTE: In Stage 3, only values of G_m need be written down, if two calculating machines are available, one to carry the value of H_m, the other to compute its reciprocal. The blank space between the ninth and tenth decimal is for the purpose of inserting the corrected value of G_m, after (λ/θ) has been determined at a later stage. The process in stage 3 starts with $G_{34} = 1/k_{34}$; $H_{32} = k_{32} - G_{34}$; the general expression in this stage is $H_m = k_m - G_{m+2}$; $G_m = 1/H_m$.

Example 1

Stage 4

Computation of B_m to Ten Significant Figures

$B_0 = H_2$	−0.62404	16221		
B_2 =	+1.00000	00000		
$B_4 = B_2 G_4$	−0.52104	06379		
$B_6 = G_6 B_4$	+ .18230	57550		
$B_8 = G_8 B_6$	− .04455	516689		
$B_{10} = G_{10} B_8$	+ .00790	57720	53	
$B_{12} = G_{12} B_{10}$	− .00105	42114	62	
$B_{14} = G_{14} B_{12}$	+ .00010	88254	839	
$B_{16} = G_{16} B_{14}$	− .00000	89181	71314	
$B_{18} = G_{18} B_{16}$	+	05926	64749	7
$B_{20} = G_{20} B_{18}$	−	00325	22710	221
$B_{22} = G_{22} B_{20}$	+	00014	96672	325
$B_{24} = G_{24} B_{22}$	−		58535	02821
$B_{26} = G_{26} B_{24}$	+		01968	23077
$B_{28} = G_{28} B_{26}$	−		00057	47744
$B_{30} = G_{30} B_{28}$	+		1	470 77
$B_{32} = G_{32} B_{30}$	−		0	033 24
$B_{34} = G_{34} B_{32}$	+		0	000 67

$2B_0^2 + B_2^2 + \cdots = 2.08562\ \ 3414 = 2(R_{r,1} + R_{r,2}) = f$ (Compare with f on the following page).

The main reason for computing the values of B_m at this stage is to obtain the quantity $(2B_0^2 + B_2^2 + \cdots)$ for purposes of checking the values of $R_{m,1}$ and $R_{m,2}$. Although it is sufficient for this purpose to carry ten decimal places in the values of B_m, they will be computed to ten significant figures, up to the maximum number of decimal places carried. Thus it will later be necessary to recompute only those values of B_m which, to nineteen decimals, are affected by the error in the characteristic value.

Example 1

Correction Process

(a) Values of $R_{m,1}$ and $R_{m,2}$ Determined from Equations (14), (16), (19), and (21).

$R_{2,1} = 0.38942\ \ 79461$
$R_{2,2} = 0.65338\ \ 37611$
$R_{4,2} = 1.12996\ \ 810390$
$R_{6,2} = 1.06164\ \ 475631$
$R_{8,2} = 1.03205\ \ 002297$
$R_{10,2} = 1.01797\ \ 220065$
$R_{12,2} = 1.01072\ \ 816427$
$R_{14,2} = 1.00674\ \ 543491$
$R_{16,2} = 1.00442\ \ 972$
$R_{18,2} = 1.00302$
$R_{20,2} = 1.002$

Check

$2(R_{2,1} + R_{2,2}) = 2B_0^2 + B_2^2 + \cdots = f$
$R_{2,1} + R_{2,2} = 1.04281\ \ 1707$

$\frac{\lambda}{\theta} = (H_{2,1} - H_{2,2})/(R_{2,1} + R_{2,2})$
$= +7.72814\ \ 42 \times 10^{-13}$
$\lambda = +1.85475\ \ 461 \times 10^{-11}$
$H_2 = -0.62404\ \ 16220\ \ 82538\ \ 1975$
See equations (22) and (23)
$\alpha = \bar{\alpha} + \lambda = -38.45897\ \ 31689\ \ 81452\ \ 4539$

Beginning with $R_{20,2} = 1 + G_{22}^2$, all others are obtained by the recurrence relation

$$R_{m,2} = 1 + G_{m+2}^2 \cdot R_{m+2,2}\,.$$

Example 1

Correction Process

(b) Correct values of G_m and H_m, obtained as indicated in (29) and (30). Usually they would be entered immediately on page 20, in the space provided for the corrected digits. They are exhibited separately here, for the sake of clarity.

$G_2 = -1.60245\ \ 72153\ \ 74227\ \ 1855$
$G_4 = -0.52104\ \ 06378\ \ 75817\ \ 45737$
$G_6 = -\ .34988\ \ 77854\ \ 53864\ \ 0723$

$H_2 = \frac{1}{2}(k_2 - G_4) = \frac{\theta}{\alpha} =$
$-0.62404\ \ 16220\ \ 82538\ \ 1974$

$G_8 = -.24439\ 80273\ 32467\ 9739$ (Compare with value on previous page.)
$G_{10} = -.17743\ 78282\ 92000\ 746$
$G_{12} = -.13334\ 70602\ 32925\ 4206$
$G_{14} = -.10322\ 92739\ 91498\ 70$
$G_{16} = -.08194\ 92916\ 01039\ 23$
$G_{18} = -.06645\ 58605\ 96783$

The underlined digits show in which decimal place the corrections begin. No corrections are necessary beyond G_{18} in the present illustration.

Example 1.

Correction Process

	(c)	Correct Values B_m				Discrepancy revealed by checking formulas
$B_0 = H_2 =$	−0.62404	16220	82538	1974		
$B_2 =$	+1.00000	00000	00000	0000		$k_2B_2 - 2B_0 - B_4 = 0$
$B_4 = G_4B_2$	−0.52104	06378	75817	4574		$k_4B_4 - B_2 - B_6 = 4 \times 10^{-19}$
$B_6 = G_6B_4$	+ .18230	57549	17838	5010		$k_6B_6 - B_4 - B_8 = 1 \times 10^{-19}$
$B_8 = G_8B_6$	− .04455	51668	73276	10171		$k_8B_8 - B_6 - B_{10} = 1 \times 10^{-19}$
$B_{10} = G_{10}B_8$	+ .00790	57720	49181	80470		$k_{10}B_{10} - B_8 - B_{12} = 5 \times 10^{-19}$
$B_{12} = G_{12}B_{10}$	− .00105	42114	61630	024337		$k_{12}B_{12} - B_{10} - B_{14} = 4 \times 10^{-20}$
$B_{14} = G_{14}B_{12}$	+ .00010	88254	83817	58410	09	$k_{14}B_{14} - B_{12} - B_{16} = 4 \times 10^{-20}$
$B_{16} = G_{16}B_{14}$	−	89181	71306	99137	5	$k_{16}B_{16} - B_{14} - B_{18} = 2 \times 10^{-20}$
$B_{18} = G_{18}B_{16}$	+	05926	64749	15564	7	$k_{18}B_{18} - B_{16} - B_{20} = 1 \times 10^{-19}$
$B_{20} = G_{20}B_{18}$	−	00325	22710	19209	79	$k_{20}B_{20} - B_{18} - B_{22} = 5 \times 10^{-20}$
$B_{22} = G_{22}B_{20}$	+	00014	96672	32345		$k_{22}B_{22} - B_{20} - B_{24} = 2 \times 10^{-19}$
$B_{24} = G_{24}B_{22}$	−	00000	58535	02814		$k_{24}B_{24} - B_{22} - B_{26} = 4 \times 10^{-19}$
$B_{26} = G_{26}B_{24}$	+		01968	23076	9	
$B_{28} = G_{28}B_{26}$	−		00057	47744	09	$k_{26}B_{26} - B_{24} - B_{28} = 6 \times 10^{-21}$
$B_{30} = G_{30}B_{28}$	+		1	47076	89	$k_{28}B_{28} - B_{26} - B_{30} = 5 \times 10^{-21}$
$B_{32} = G_{32}B_{30}$	−			03323	85	$k_{30}B_{30} - B_{28} - B_{32} = 2 \times 10^{-21}$
$B_{34} = G_{34}B_{32}$	+			00066	8	$k_{32}B_{32} - B_{30} - B_{34} = 2 \times 10^{-21}$

The largest discrepancies seem to be on the lines alongside B_4 and B_{10}. Since k_4 is approximately -2.3 and the error in B_4 is multiplied by this amount in the checking formula ($k_4B_4 - B_2 - B_6$), the actual error in B_4 may be about 2×10^{-19}. For similar reasons, the uncertainty in the other coefficients is no more than 2×10^{-19}.

The coefficients B_m may now be normalized in any required manner. Since the process is perfectly straightforward, it will not be exhibited.

Example 2. Let it be required to compute Fourier coefficients corresponding to the fifth even characteristic value, for $\theta = 21$. The corresponding value of α, to nine decimals, is 37.462613226. We shall aim at values of B_m to 12 decimal places, so as to be able to obtain the normalized coefficients (in Ince's form or that of Stratton-Morse-Chu-Hutner) to ten decimal places. Some subsidiary computations will be carried to a higher accuracy.

Stage 1. Computation of k_m

Values of k_m based on $\bar{\alpha} = 37.462613226$	Values of k_m based on the corrected value of $\alpha = 37.46261\ 32260\ 28197$
$k_1 = 1.73631\ 49155\ 23809\ 524$	$k_1 = 1.73631\ 49155\ 25152$
$k_3 = 1.35536\ 25345\ 71428\ 571$	72771
$k_5 = 0.59345\ 77726\ 66666\ 667$	68009
$k_7 = -0.54939\ 93701\ 90476\ 190$	89133
$k_9 = -2.07320\ 88940\ 00000$	$39\ 98657$
$k_{11} = -3.97797\ 07987\ 61905$	60562
$k_{13} = -6.26368\ 50844\ 76190$	7485
$k_{15} = -8.93035\ 17511\ 42857$	415
$k_{17} = -11.97797\ 07987\ 61905$	The omitted digits are the same as in the uncorrected values of k_m at the left.
$k_{19} = -15.40654\ 22273$	
$k_{21} = -19.21606\ 60369$	The corrected value of α was obtained after stage 4, in the "Correction Process."
$k_{23} = -23.40654\ 22273$	
$k_{25} = -27.97797\ 07988$	
$k_{27} = -32.930$	

$$k_m = (\alpha - m^2)/\theta$$

Example 2

Stage 2 and **Stage 3.** Computation of G_m and H_m

Stage 2

$G_{3,1} = 0.73631\ 49155\ 23809\ 524$
$= k_1 - 1$

$G_{5,1} = -.00275\ 20152\ 18190\ 54$
$= k_3 - H_{3,1}$

$G_{7,1} = +363.96354\ 14081\ 43$
$= k_5 - H_{5,1}$

$H_{3,1}$ 1.35811 45497 89619 11
$= 1/G_{3,1}$

$H_{5,1} = -363.37008\ 36354\ 76$
$= 1/G_{5,1}$

$H_{7,1} = +0.00274\ 75279\ 42307\ 3$
$= 1/G_{7,1}$

$H_{7,2} = +0.00274\ 75279\ 36719\ 6$

Stage 3

$G_{9,2} = -0.55214\ 68981\ 271958$
$G_{11,2} = -0.26209\ 66639\ 18008$
$G_{13,2} = -0.16258\ 45780\ 77957$
$G_{15,2} = -0.11304\ 02212\ 19115$
$G_{17,2} = -0.08394\ 30196\ 73521$
$G_{19,2} = -0.06512\ 79693\ 129$
$G_{21,2} = -0.05215\ 59237\ 0$
$G_{23,2} = -0.04278\ 85084$

$G_{25,2} = -0.03578\ \ 13$
$G_{27,2} = -0.0304$

Two calculating machines were used in computing the quantities in stage 3; one to hold the quantity $H_{m,2} = k_m - G_{m+2,2}$, the other to obtain $G_{m,2} = 1/H_{m,2}$. The process started with $G_{27,2} = 1/k_{27}$.

Example 2

Stage 4. Computation of B_m to Ten Significant Figures

(The first eight values of B_m have been computed to more than ten significant figures. See page 19 for reason.)

$B_1 = H_3B_3 = -1.35590\ \ 00872\ \ 291$
$B_3 = H_5B_5 = -0.99836\ \ 94581\ \ 869$
$B_5 = H_7 = +\ .00274\ \ 75279\ \ 42307$
$B_7 = +1.00000\ \ 00000\ \ 00000$
$B_9 = G_9 = -\ .55214\ \ 68981\ \ 272$
$B_{11} = G_{11}B_9 = +\ .14471\ \ 58599\ \ 918$
$B_{13} = G_{13}B_{11} = -\ .02352\ \ 85670\ \ 3796$
$B_{15} = G_{15}B_{13} = +\ .00265\ \ 96744\ \ 2294$
$B_{17} = G_{17}B_{15} = -\ .00022\ \ 32611\ \ 024$
$B_{19} = G_{19}B_{17} = +\ .00001\ \ 45405\ \ 4223$
$B_{21} = G_{21}B_{19} = -\ \ \ \ \ \ 07583\ \ 75411$
$B_{23} = G_{23}B_{21} = +\ \ \ \ \ \ 00324\ \ 49753$
$B_{25} = G_{25}B_{23} = -\ \ \ \ \ \ 00011\ \ 61094$
$B_{27} = G_{27}B_{25} = +\ \ \ \ \ \ \ \ \ \ \ \ 353$

$$\sum B_m^2 = 4.16158\ \ 3764 = f$$

Check $R_{7,1} + R_{7,2}$ on the following page.

Example 2

Correction Progress

Computation of $R_{m,1}$ and $R_{m,2}$, H_7, and λ/θ

$R_{3,1} = 1.84447\ \ 5 = H_3^2$
$R_{5,1} = 3.75578\ \ 3 = (1 + R_{3,1})H_5^2$
$R_{7,1} = 2.83521\ \ 42 = (1 + R_{5,1})H_7^2$
$R_{7,2} = 1.32636\ \ 96 = (1 + G_9^2R_9)$
$R_{9,2} = 1.07053\ \ \ \ \ = (1 + G_{11}^2R_{11})$
$R_{11,2} = 1.02677\ \ \ \ \ = (1 + G_{13}^2R_{13})$
$R_{13,2} = 1.0128\ \ \ \ \ \ = (1 + G_{15}^2)$
$R_{7,1} + R_{7,2} = 4.1615838$ (check f on previous page)
$H_7 = (R_{7,2}H_{7,1} + R_{7,1}H_{7,2})/(R_{7,1} + R_{7,2}) = 0.00274\ \ 75279\ \ 385005$
$\lambda/\theta = (H_{7,1} - H_{7,2})/(R_{7,1} + R_{7,2}) = +1.3427 \times 10^{-12}$
$\lambda = +2.81967 \times 10^{-11}$; $\alpha = \bar{\alpha} + \lambda = 37.46261\ \ 32260\ \ 28197$

Example 2

Corrected Values of G_m and H_m

$$H_3 = H_{3,1} - \frac{\lambda}{\theta} R_{3,1} = 1.35811\ \ 45497\ \ 8\underline{7}14$$

$$H_5 = H_{5,1} - \frac{\lambda}{\theta} R_{5,1} = -363.37008\ \ \underline{4}13976$$

$$H_7 = H_{7,1} - \frac{\lambda}{\theta} R_{7,1} = +0.00274\ \ 75279\ \ 3\underline{8}5005$$

$$= H_{7,2} + \frac{\lambda}{\theta} R_{7,2}$$ (Compare with value on previous page.)

$$G_9 = G_{9,2} - \frac{\lambda}{\theta}(R_{7,2} - 1) = -0.55214\ \ 68981\ \ 27\underline{6}34$$

$$G_{11} = G_{11,2} - \frac{\lambda}{\theta}(R_{9,2} - 1) = -0.26209\ \ 66639\ \ 18\underline{1}03$$

No corrections are necessary in the other values of G_m.

Values of B_m Based on Corrected Values of G_m and H_m

Values of B_m		Discrepancy revealed by checking formulas
$B_1 = H_3 B_3$	$= -1.35590\ \ 00872\ \ 29\underline{9}4$	$k_3 B_3 - B_5 - B_1 = 0$
$B_3 = H_5 B_5$	$= -0.99836\ \ 94581\ \ 8\underline{9}267$	$k_5 B_5 - B_7 - B_3 = 7 \times 10^{-15}$
$B_5 = H_7$	$= +0.00274\ \ 75279\ \ 3\underline{8}5005$	$k_7 B_7 - B_5 - B_9 = 0$
$B_7 =$	$+1.00000\ \ 00000\ \ 00000$	$k_9 B_9 - B_7 - B_{11} = 6 \times 10^{-15}$
$B_9 = G_9$	$= -0.55214\ \ 68981\ \ 27\underline{6}34$	$k_{11} B_{11} - B_9 - B_{13} = 2 \times 10^{-14}$
$B_{11} = B_9 G_{11}$	$= +0.14471\ \ 58599\ \ 9\underline{1}981$	$k_{13} B_{13} - B_{11} - B_{15} = 2 \times 10^{-13}$
See values B_{13}, B_{15}, etc. derived in Stage 4.		$k_{17} B_{17} - B_{15} - B_{19} = 10^{-13}$
		$k_{21} B_{21} - B_{19} - B_{23} = 3 \times 10^{-15}$
		$k_{25} B_{25} - B_{23} - B_{27} = 0$

The underlined digits indicate the first decimal place which differs from the uncorrected values.

In stage 4, the first eight values of B_m were computed to more than ten significant figures, for purposes of illustrating an interesting phenomenon. Comparing the corrected values of H_m with those obtained in stage 2, it is noticed that $H_{7,1}$ and $H_{5,1}$ are in error by 4 and 5 units, respectively, in the tenth significant figure. Hence one would expect that B_3 (in stage 4) which is equal to $H_7 H_5$, would be correct to at most ten significant figures; actually B_3 is incorrect by two units in the twelfth significant figure, The reason for this may be deduced from the following analysis of the error in B_m due to an error in α.

Let δ_3 be the error in B_3, due to an error in α, and let $B_{3,1} = H_{7,1} H_{5,1}$. Hence from (14) and (15)

$$\delta_3 \cong \eta_{7,1} H_{5,1} + \eta_{5,1} H_{7,1}$$

$$= -\frac{\lambda}{\theta}\{(H_7^2 + H_7^2H_6^2 + H_7^2H_6^2H_5^2)H_6 + (H_6^2 + H_6^2H_5^2)H_7\}$$

$$= -\frac{\lambda}{\theta}(H_6H_7)(H_7 + H_7H_6^2 + H_7H_6^2H_5^2 + H_6 + H_6H_5^2)$$

$$= -\frac{\lambda}{\theta}H_6H_7[(H_6 + H_6H_5^2)(H_7H_6 + 1) + H_7]$$

Although H_6 is numerically large, H_6H_7 is negative and numerically slightly less than one—and $(H_7H_6 + 1)$ is approximately equal to 0.0016; consequently $(H_6 + H_6H_5^2)(1 + H_7H_6)$ is no greater numerically than 1.7. For this reason, the error in B_3, due to an error in α, is numerically less than $2\lambda/\theta$. Thus if enough significant figures are kept in H_m and B_m to make the rounding-off error small, the error in the quantity B_m, due to an error in α, is much less than might be expected on the basis of the accuracy of the ratios H_m. This situation seems to hold in many other cases, in the region where one of the ratios H_m (or G_m) becomes numerically large.

It may be worth while to make one more observation. In generating the values of G_m of stages 2 and 3, the operator notices that $G_{3,1}$ is less than unity. Consequently, he would stop at this point after computing $H_{3,1}$ and begin generating the values of stage 3, beginning with $G_{27,2}$ and continuing until he reached $H_{7,2}$ which is numerically less than unity. He then has two alternatives:

(a) To go back to stage 2 and generate values up to $H_{7,1}$. This was done in the present illustration.

(b) To keep on generating values by the formulas of stage 3, ending the process with $H_{3,1}$.

If the second alternative had been adopted, the results would have been just as satifactory. As a matter of fact, the following rule always leads to good results.

(a) If G_3 (or G_2) is less than one, end stage 2 after computing H_3 (or H_2); if G_3 is greater than one, generate $G_3, \cdots, G_r$ (or $G_2 \cdots G_r$) where $G_r \geq 1$, $G_{r+2} < 1$. Stage 2 ends with H_r.

If some G_n is much larger numerically than the rest, then several significant figures may be lost in generating $G_{n\pm2}$; consequently the values of k_m, G_m, and H_m should be computed to several additional significant figures in such a region.

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TENSORS AND EQUIVALENT CIRCUITS

By Banesh Hoffmann

§1. In various articles dealing with the setting up of equivalent electrical networks for mechanical, electrodynamical, and other systems, including field systems, Kron has shown how he was guided to the discovery of the appropriate equivalent circuit by making use of tensor concepts. He has stated that only tensors can be represented by physical models, and has employed this as a principle in setting up his equivalent circuits.

Wishing to describe in the simplest possible terms his method of applying this principle in a particular case, Kron has written[1]

"... if the standard hunting equations are expressed as

$$A + B = 0$$

(where none of the sets of expressions A or B nor the combination $A + B$ are tensors), the tensorial hunting equations may be expressed as

$$A + B + C - C = 0$$

where C represents a set of expressions that otherwise would cancel.

Now while neither A nor C are tensors, the combination $A + C$ is a tensor, so is the combination $B - C$. Hence each expression [meaning each total expression enclosed in parentheses] in the equation

$$(A + C) + (B - C) = 0$$

is a tensor."

He has amplified this slightly in another paper[2] where in addition to the above he remarks that in $A + B = 0$ the 0 was not a tensor, but that in $(A + C) + (B - C) = 0$ the 0 was a tensor, since "it had acquired a different law of transformation by the appearance of C."

It is clear that these statements cannot be correct. If $(A + B)$ is not a tensor, then $\{(A + C) + (B - C)\}$, being identical with it, cannot be a tensor. If the one is a tensor the other must be also, and if the one is not then the other cannot be either. On account of the equality signs, similar remarks must also apply to the two 0's. A reshuffling of terms on the left hand side cannot of itself bring in a new law of transformation. If, instead of starting with $(A + B)$ we start with $\{(A + C) + (B - C)\}$, it becomes clear that Kron is, in effect, implying that a mere algebraic cancellation of terms can change a tensor into a non-tensor, though he himself is certainly aware that this cannot be so.

Since the $(A + B)$ statements are false, and since they purport to outline the main work of E.C.O.S., the validity of that paper as a whole is apparently called into question.

[1] Equivalent Circuits for Oscillating Systems and the Riemann-Christoffel Curvature Tensor, A.I.E.E. Trans., 62, p. 25 (1943). This paper will be referred to as E.C.O.S.

[2] Equivalent Circuits of the Field Equations of Maxwell, Proc. I.R.E. 32, p. 289 (1944).

It is the purpose of the present paper to show that Kron's $(A + B)$ statements do not properly or fairly describe the procedures used by him in setting up the equivalent circuits to which they refer; that they therefore in no way invalidate the essential parts of the paper in question; and that the real tensorial significance of the methods used by Kron in E.C.O.S. lies somewhat deeper than is indicated by his statements quoted above.

The present paper purposely avoids any discussion of the problems posed by Kron's statement that only tensors can be represented by physical models.

§2. If the method used by Kron in obtaining the equivalent circuits in E.C.O.S. faithfully follows the outline he gives, it must be incorrect in some particular. On the other hand, if the method deviates from the outline in an essential manner there opens up the possibility that the method is valid and that the outline can be suitably modified to take account of the deviation.

The key to the present situation lies in the fact that we are dealing with a problem in small oscillations about a steady state of motion. For the sake of simplicity let us denote the equations of motion formally by the tensor equation

$$V_\alpha = 0. \tag{1}$$

The non-tensorial equations of small oscillations will then take the form

$$\Delta V_\alpha = 0, \tag{2}$$

where Δ denotes the ordinary and not the covariant differential.

In order to obtain a tensor equation from (1) by differentiation, it is necessary to use the covariant derivative. Denoting the covariant differential by δ, we may write the tensor equation obtained by differentiating (1) covariantly in the form

$$\delta V_\alpha = 0, \tag{3}$$

or, on expanding the covariant differential, in the form

$$\Delta V_\alpha - \Gamma^\gamma_{\alpha\beta} V_\gamma \Delta x^\beta = 0, \tag{4}$$

the quantity $\Gamma^\gamma_{\alpha\beta}$ being the connection with respect to which the covariant derivative is made.

Provided (4) is taken in conjunction with the original equations of motion (1), the second term in (4) can be removed, so that (4) and (2) will coincide. In this sense, then, (2) is a tensor equation. That is, (2) is not a tensor equation if taken by itself, but the combination of (1) with (2), being equivalent to the combination of (1) with (4), is equivalent to a pair of tensor equations.

The standard method of attack on problems of small oscillations about a steady state of motion is to obtain as an initial approximation a steady state solution of the equations of motion (1), to substitute these solutions into the equations of small oscillations, and then to solve these latter equations for the quantities giving the first order deviations from the steady state solution. The substitution of a steady state solution of (1) into (4) will cause the second term in (4) to become

zero, and thus will reduce (4) to the equivalent of (2). Therefore so far as the usual method of solution by successive approximation is concerned it is permissible to treat (4) as the equivalent of (2), since (1) is automatically used in conjunction with either (2) or (4).

While this ensures that the small oscillation solutions obtained from (4) would be the same as those obtained from (2), it does not quite cover the present situation. For in the analysis preliminary to setting up his equivalent circuits Kron does not make an actual substitution of a steady state solution into the equations of small oscillations. Instead he sets up an equivalent circuit for the steady state solution and another for the small oscillations, and only after he has devised these circuits does he obtain values from the first in order to place them on the second. The $(A + B)$ statements refer to what is done in order to devise the equivalent circuits in the first place rather than to what is done after they have been set up. Nevertheless it is evident from the above, and is of course quite well known, that in the standard method of solution of problems in small oscillations it is permissible to use the original equations of motion in order to modify or simplify the equations of small oscillations. And, as has been pointed out, this would allow the non-tensor equations of small oscillations (2) to be converted into the tensor equation (4) in a legitimate manner. Let us therefore examine Kron's paper carefully in order to see whether a conversion of this type is actually made therein.

In E.C.O.S. just below equation (10) it is stated that the quantity ΔE_α defined in (10) is simplified by means of the original equations of motion (11) to bring it to the form (12). At first sight this appears to be a conversion of the type discussed above. But a closer examination reveals that this simplification by means of the original equations of motion does not cause the cancellation of unlike terms which would not otherwise cancel. Though it is indeed an insertion of the original equations of motion it is not an insertion of the type needed for the removal of the $(A + B)$ difficulty in the manner suggested above. Despite this it will be seen later to be not without significance for the problem in question.

A closer examination of the details of Kron's paper brings to light an instance of true cancellation of unlike terms by means of the original equations of motion, but it is hidden away among the calculations rather than presented explicitly.[3] In the third row on the right of equation (9) in E.C.O.S. there appear the two terms

$$\Gamma_{\gamma\delta,\alpha}\, e^{\gamma}\, \Delta x^{\delta} - \Gamma_{\gamma\delta,\alpha} R^{\gamma}_{.\beta}\, i^{\beta}\, \Delta x^{\delta},$$

which we shall denote for brevity by the symbol Q. These terms are supposed to arise from the expansion of the quantity $a_{\alpha\beta}\delta(\delta i^{\beta})/dt$ appearing to their left.

[3] The equations concerned were quoted by Kron from a much earlier publication by him in which the substitution was also only implicit. Since in the earlier paper there was no special reason to make an explicit announcement that the substitution had been made, it remained hidden. Meanwhile, however, Kron's emphasis on the role of tensors in his method, and his (A + B) outline of it, brought to the hidden substitution a new significance which made necessary its present disinterment.

But it is clear that the expansion of this quantity, which contains only the metrical tensor, the connection, and the currents, could not of itself involve the voltage vector e^γ or the resistance tensor $R^\gamma_{.\beta}$. It will be noted too that, though the terms Q do not find a place in the routine expansion of the quantity $a_{\alpha\beta}\delta(\delta i^\beta)/dt$, there is in that routine expansion another quantity

$$P \equiv \Gamma_{\gamma\delta,\alpha}\,(\delta i^\gamma/dt)\,\Delta x^\delta$$

which is missing from the expansion as given in equation (9) of E.C.O.S.

Moreover, it happens that P may be replaced by Q by virtue of the original equations of motion. For we have

$$P - Q = \Gamma_{\gamma\delta,\alpha}\,\Delta x^\delta\,\{\delta i^\gamma/dt + R^\gamma_{.\beta}\,i^\beta - e^\gamma\} \tag{5}$$

and the expression in parentheses in (5), when set equal to zero, yields the original equations of motion. Thus a cancellation of unlike terms has in fact been performed with the aid of the original equations of motion.

In the notation of equations (1) to (4) of this paper, we have

$$P - Q = \Gamma_{\gamma\delta,\alpha}\,\Delta x^\delta\,V^\alpha = \Gamma^\alpha_{\gamma\delta}\,\Delta x^\delta\,V_\alpha\,, \tag{6}$$

showing that this cancellation is of just the sort required by the argument based on equations (1) to (4) above.

Now, instead of replacing the quantity $a_{\alpha\beta}\delta(\delta i^\beta)/dt$ by the terms Q, Kron could have achieved an even simpler cancellation by replacing the term $\Gamma_{\gamma\delta,\alpha}e^\gamma\Delta x^\delta$, which appears in the first row of his equation (9), by the terms

$$\Gamma_{\gamma\delta,\alpha}\,\Delta x^\delta\,\{R^\gamma_{.\beta}\,i^\beta + \delta i^\gamma/dt\}$$

which are equal to it according to the original equations of motion.

In the light of this remark, let us return to Kron's simplification of his equation (10) into (12) by means of the original equations of motion. This simplification and the previous $P \rightarrow Q$ cancellation, taken together, are seen to be equivalent to the single substitution suggested above for the term $\Gamma_{\gamma\delta,\alpha}e^\gamma\Delta x^\delta$. Essentially, therefore, Kron has made that single substitution.

Kron's brief $(A + B)$ outline can now be easily amended to give a proper account of the process used by him in setting up the equivalent circuits for oscillating systems.

The original quantity $(A + B)$ is not a tensor in its own right, but it can be made into a tensor with the aid of the original equations of motion. The equation

$$A + B = 0 \tag{7}$$

is not rewritten in the form

$$(A + C) + (B - C) = 0 \tag{8}$$

but in the form

$$(A + C) + (B - C') = 0 \tag{9}$$

where, though C and C' are not identically equal, they do satisfy the relation

$$C - C' = 0 \tag{10}$$

as a direct algebraic consequence of the original equations of motion. The quantities $(A + C)$ and $(B - C')$ entering (9) are tensors, and equation (9) is a tensor equation, though equation (7) is not, and neither is (8).

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AUXILIARY TABLE OF COMPLETE ELLIPTIC INTEGRALS

By E. L. Kaplan

The integrals in question are $K = \int_0^{\pi/2} (1 - k^2 \sin^2\varphi)^{-1/2}\, d\varphi$ and $E = \int_0^{\pi/2} (1 - k^2 \sin^2\varphi)^{1/2}\, d\varphi$. The present table gives ten-decimal values of K and E as functions of $\log_{10} k'^2 = -\lambda$ for $\lambda = 1(.005)2(.01)6$, where $k'^2 = 1 - k^2$. Larger values of λ will rarely be encountered, but if they are, the formulas

$$K \doteq (1 + \tfrac{1}{4}k'^2) \ln \frac{4}{k'} - \tfrac{1}{4}k'^2$$

$$E \doteq \tfrac{1}{2}k'^2 \ln \frac{4}{k'} + 1 - \tfrac{1}{4}k'^2$$

give ten-decimal accuracy.

The only comparable table known to the writer which facilitates the getting of values of K and E in the neighborhood of their singularity is that of Airey (1). He has tabulated to ten decimals functions K_1, K_2, E_1, E_2, such that

$$K = K_1 \ln \frac{4}{k'} - K_2, \qquad E = E_1 \ln \frac{4}{k'} + E_2$$

For each of K and E there are thus three quantities to be obtained from tables, and some arithmetical operations to be done besides. The importance of these functions makes it worthwhile to provide a more direct means of getting their values for small values of k'. With the present tables, the only preliminary to extracting the value desired is the determination of $\log_{10} k'^2$. The number k'^2 is often a simple fraction, whose logarithm is easily obtained without interpolation as the difference of two tabular logarithms. The tabular argument is given as a logarithm with a negative integral part and a positive decimal, in order that the user of the table may be spared the trouble of getting the cologarithm when k'^2 is given as a decimal.

The method of the arithmetico-geometric mean, followed by subtabulation to one-tenth of the original interval, was used for $1 \leqq \lambda \leqq 2$, while the rest of the table was computed by means of auxiliary functions like Airey's. In both cases the check was by differencing. The use of both methods for $\lambda = 2$ is a check on errors of procedure. From eleven to fourteen decimals were retained.

The largest error resulting from a neglect of second differences is 1.0×10^{-6}; from a neglect of third differences, 5×10^{-10}, provided that the nearest mean second difference is employed: $\mu\delta^2 u_{1/2} = \tfrac{1}{2}(u_{-1} - u_0 - u_1 + u_2)$. For typographical reasons these differences have been dropped down half a line, so that $\mu\delta^2 u_{1/2}$ is aligned with u_1. If ordinary second differences are used, and third differences neglected, the maximum error is 4×10^{-9}, nearly eight times as

great. The differences supplied are those required by the following interpolation formulas, which give the full accuracy of the table:

For $1 \leqq \lambda \leqq 2$, and $0 \leqq \theta \leqq .5$, $f(\Lambda + .01\theta)$

$$= 2\left\{\theta \cdot f(\Lambda + .005) + (.5 - \theta) \cdot f(\Lambda) - \theta(.5 - \theta) \cdot \delta^2 f\left(\Lambda + \frac{.5 + \theta}{300}\right)\right\}$$

For $2 \leqq \lambda \leqq 6$, and $0 \leqq \theta \leqq 1$, $f(\Lambda + .01\theta)$

$$= \theta \cdot f(\Lambda + .01) + (1 - \theta) \cdot f(\Lambda) - \theta(1 - \theta) \cdot \tfrac{1}{2}\delta^2 f\left(\Lambda + \frac{1 + \theta}{300}\right)$$

Here Λ is a tabular argument, a multiple of .005 or .01 respectively, and the indicated second differences (if required) are obtained by a rough linear interpolation (such as is easily done mentally) between the given values of $\mu\delta^2 f$, which correspond to arguments $\Lambda \pm .0025$ or $\Lambda \pm .005$, respectively. These formulas are equivalent to Everett's formula using a pair of second differences, and are simpler where, as here, the third differences are small.

The most useful general tables of K and E are those of Hayashi (2), who gives ten-decimal values for $k^2 = 0(.001)1$. When k^2 is near unity, however, it is difficult or impossible to interpolate in these tables—hence the need for some kind of auxiliary table. Since Hayashi's tables in general are notorious for their inaccuracy, the writer checked these by differencing (generally every third one of the fourth differences) either the values themselves, or auxiliary functions calculated therefrom. They were also compared with Milne-Thomson's (3) nine-decimal table for $k^2 = 0(.01)1$, which served only to show that the last figure in Milne-Thomson's table is unreliable. The following errors were found in Hayashi's table (only the four starred values were actually recomputed):

	K				*E*		
k^2	*Decimals*	*For*	*Read*	k^2	*Decimals*	*For*	*Read*
.999	8–9	06	60*	.052	9–10	45	55*
.013	8–9	68	86	.939	9	4	3
.095	0	0	1	.936	9	9	8
.737	9–10	56	65*	.201	3–4	65	86
.493	9	4	9	.732	10	6	8*
				.668	0	0	1

.669—The first two digits of the argument should be given.

The eleventh and twelfth decimals given in a portion of the table were not checked, nor were any of the rather inadequate auxiliary tables.

Famous tables of elliptic integrals, both complete and incomplete, to 9 or more decimals, as functions of $\theta = \sin^{-1} k$, are those of Legendre (4). An interesting set of six-place tables by Heuman (5) includes a list of about 40 errata in

Legendre's tables. The 5–7 decimal tables of Samoilova-Yakhontova (6) make the change from Legendre's θ to k^2, and list errata likewise.

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$\log_{10}k'^2$	K	$\mu\delta^2$	E	$\mu\delta^2$	$\log_{10}k'^2$	K	$\mu\delta^2$	E	$\mu\delta^2$
	2.		1.			2.		1.	
1.000	5780921133		1047747327		2.800	7966672065	19776	0728514026	58413
2.995	5835106424	25544	1038363718	79561	.795	8021757497	19641	0721860346	57943
.990	5889317177	25383	1029059360	78944	.790	8076862504	19507	0715264376	57477
.985	5943553233	25223	1019833638	78331	.785	8131986951	19374	0708725650	57014
.980	5997814433	25064	1010685943	77723	.780	8187130706	19242	0702243708	56555
2.975	6052100618	24906	1001615668	77119	2.775	8242293636	19110	0695818093	56099
.970	6106411629	24748	0992622210	76519	.770	8297475610	18978	0689448349	55646
.965	6160747309	24591	0983704973	75923	.765	8352676497	18848	0683134025	55196
.960	6215107502	24434	0974863361	75331	.760	8407896167	18718	0676874674	54750
.955	6269492050	24278	0966096785	74743	.755	8463134490	18589	0670669851	54307
2.950	6323900798	24123	0957404660	74159	2.750	8518391337	18460	0664519114	53867
.945	6378333593	23968	0948786403	73580	.745	8573666580	18332	0658422025	53430
.940	6432790278	23814	0940241437	73004	.740	8628960092	18205	0652378149	52997
.935	6487270701	23661	0931769188	72432	.735	8684271745	18078	0646387053	52566
.930	6541774709	23509	0923369086	71864	.730	8739601414	17952	0640448310	52139
2.925	6596302149	23357	0915040565	71300	2.725	8794948972	17827	0634561492	51715
.920	6650852871	23206	0906783063	70740	.720	8850314295	17703	0628726178	51294
.915	6705426722	23055	0898596022	70184	.715	8905697258	17579	0622941948	50876
.910	6760023554	22905	0890478888	69632	.710	8961097739	17455	0617208386	50461
.905	6814643215	22756	0882431110	69083	.705	9016515613	17333	0611525077	50049
2.900	6869285558	22607	0874452142	68538	2.700	9071950758	17211	0605891612	49640
.895	6923950434	22459	0866541441	67997	.695	9127403054	17089	0600307584	49234
.890	6978637695	22312	0858698468	67460	.690	9182872378	16969	0594772587	48831
.885	7033347195	22165	0850922687	66927	.685	9238358611	16849	0589286220	48431
.880	7088078787	22020	0843213568	66397	.680	9293861632	16729	0583848085	48034
2.875	7142832326	21874	0835570582	65871	2.675	9349381323	16610	0578457787	47640
.870	7197607607	21730	0827993205	65349	.670	9404917566	16492	0573114932	47249
.865	7252404665	21586	0820480917	64830	.665	9460470242	16375	0567819131	46860
.860	7307223178	21443	0813033201	64316	.660	9516039234	16258	0562569997	46475
.855	7362063061	21300	0805649544	63804	.655	9571624426	16142	0557367145	46092
2.850	7416924173	21158	0798329437	63296	2.650	9627225702	16026	0552210196	45713
.845	7471806372	21017	0791072373	62792	.645	9682842947	15912	0547098770	45336
.840	7526709518	20876	0783877850	62292	.640	9738476047	15797	0542032492	44962
.835	7581633470	20736	0776745370	61795	.635	9794124886	15684	0537010989	44590
.830	7636578088	20597	0769674436	61301	.630	9849789353	15571	0532033891	44222
2.825	7691543234	20459	0762664558	60811	2.625	9905469334	15458	0527100831	43856
.820	7746528770	20321	0755715246	60325	.620	9961164718	15347	0522211445	43493
.815	7801534557	20183	0748826017	59841	.615	*0016875392	15236	0517365370	43132
.810	7856560459	20047	0741996389	59362	.610	*0072601247	15125	0512562248	42775
.805	7911606340	19911	0735225883	58886	.605	*0128342172	15015	0507801723	42420

$\log_{10}k'^2$	K	$\mu\delta^2$	E	$\mu\delta^2$	$\log_{10}k'^2$	K	$\mu\delta^2$	E	$\mu\delta^2$
	3.		1.			3.		1.	
2.60	0184098058	14906	0503083440	42067	2.40	2425419972	11034	0345242854	29972
.595	0239868796	14798	0498407049	41717	.395	2481702494	10949	0341982768	29715
.59	0295654277	14690	0493772201	41370	.39	2537995922	10864	0338752270	29461
.585	0351454394	14582	0489178550	41026	.385	2594300172	10780	0335551106	29208
.58	0407269040	14476	0484625755	40684	.38	2650615160	10697	0332379023	28957
2.575	0463098108	14370	0480113472	40345	2.375	2706940803	10614	0329235773	28708
.57	0518941494	14264	0475641366	40008	.37	2763277020	10532	0326121107	28461
.565	0574799091	14159	0471209100	39674	.365	2819623727	10450	0323034779	28216
.56	0630670795	14055	0466816342	39342	.36	2875980843	10369	0319976546	27974
.555	0686556502	13952	0462462760	39013	.355	2932348287	10288	0316946165	27732
2.55	0742456110	13849	0458148028	38687	2.35	2988725979	10208	0313943397	27493
.545	0798369514	13746	0453871819	38362	.345	3045113839	10128	0310968003	27256
.54	0854296614	13644	0449633812	38041	.34	3101511787	10049	0308019747	27021
.535	0910237308	13543	0445433685	37721	.335	3157919745	9970	0305098394	26787
.53	0966191494	13443	0441271120	37405	.33	3214337634	9892	0302203713	26556
2.525	1022159073	13343	0437145803	37090	2.325	3270765377	9815	0299335472	26326
.52	1078139945	13243	0433057419	36778	.32	3327202896	9737	0296493443	26098
.515	1134134010	13145	0429005657	36469	.315	3383650115	9661	0293677398	25872
.51	1190141172	13046	0424990211	36161	.31	3440106956	9585	0290887113	25648
.505	1246161330	12949	0421010773	35857	.305	3496573345	9510	0288122364	25425
2.50	1302194389	12852	0417067039	35554	2.30	3553049206	9435	0285382929	25204
.495	1358240251	12755	0413158709	35254	.295	3609534465	9360	0282668588	24985
.49	1414298821	12660	0409285484	34956	.29	3666029047	9286	0279979124	24768
.485	1470370003	12564	0405447066	34660	.285	3722532878	9213	0277314320	24553
.48	1526453701	12470	0401643161	34367	.28	3779045885	9140	0274673962	24339
2.475	1582549823	12376	0397873478	34076	2.275	3835567996	9067	0272057836	24127
.47	1638658273	12282	0394137726	33787	.27	3892099137	8995	0269465732	23917
.465	1694778959	12189	0390435617	33501	.265	3948639239	8924	0266897440	23708
.46	1750911788	12097	0386766866	33216	.26	4005188228	8853	0264352752	23501
.455	1807056668	12005	0383131190	32934	.255	4061746035	8782	0261831463	23296
2.45	1863213508	11914	0379528307	32654	2.25	4118312589	8712	0259333368	23093
.445	1919382216	11824	0375957940	32377	.245	4174887820	8643	0256858263	22891
.44	1975562703	11734	0372419810	32101	.24	4231471659	8573	0254405949	22690
.435	2031754879	11644	0368913645	31827	.235	4288064037	8505	0251976226	22492
.43	2087958654	11555	0365439171	31556	.23	4344664885	8437	0249568895	22295
2.425	2144173940	11467	0361996117	31287	2.225	4401274137	8369	0247183760	22099
.42	2200400649	11379	0358584217	31020	.22	4457891723	8302	0244820628	21905
.415	2256638693	11292	0355203203	30755	.215	4514517578	8235	0242479304	21713
.41	2312887986	11205	0351852812	30492	.21	4571151635	8169	0240159597	21522
.405	2369148441	11119	0348532783	30231	.205	4627793828	8103	0237861318	21333

$\log_{10}k'^2$	K	$\mu\delta^2$	E	$\mu\delta^2$	$\log_{10}k'^2$	K	$\mu\delta^2$	E	$\mu\delta^2$
	3.		1.			3.		1.	
$\bar{2}.20$	4684444090	8038	0235584277	21145	$\bar{2}.10$	5819047067	6824	0194226783	17701
.195	4741102358	7973	0233328289	20959	.095	5875852446	6767	0192354871	17543
.19	4797768566	7908	0231093167	20775	.09	5932664563	6711	0190500424	17387
.185	4854442651	7844	0228878729	20592	.085	5989483365	6656	0188663286	17232
.18	4911124547	7781	0226684791	20410	.08	6046308795	6601	0186843302	17078
$\bar{2}.175$	4967814193	7718	0224511172	20230	$\bar{2}.075$	6103140799	6546	0185040319	16925
.17	5024511525	7655	0222357695	20052	.07	6159979321	6492	0183254186	16774
.165	5081216481	7593	0220224180	19874	.065	6216824310	6438	0181484752	16624
.16	5137928999	7531	0218110451	19699	.06	6273675709	6385	0179731867	16476
.155	5194649017	7470	0216016334	19525	.055	6330533468	6332	0177995384	16328
$\bar{2}.15$	5251376474	7409	0213941654	19352	$\bar{2}.05$	6387397531	6279	0176275155	16182
.145	5308111309	7348	0211886241	19180	.045	6444267848	6227	0174571036	16037
.14	5364853463	7288	0209849922	19011	.04	6501144366	6175	0172882882	15893
.135	5421602876	7229	0207832530	18842	.035	6558027034	6124	0171210549	15751
.13	5478359487	7170	0205833895	18675	.03	6614915799	6073	0169553896	15609
$\bar{2}.125$	5535123239	7111	0203853852	18509	$\bar{2}.025$	6671810612	6022	0167912781	15469
.12	5591894073	7053	0201892236	18345	.02	6728711422	5972	0166287067	15330
.115	5648671930	6995	0199948883	18182	.015	6785618178	5922	0164676613	15192
.11	5705456752	6937	0198023630	18020	.01	6842530831	5872	0163081283	15056
.105	5762248484	6880	0196116317	17860	.005	6899449331	5823	0161500940	14920
					2.00	6956373630	5774	0159935450	14786

$\log_{10}k'^2$	K	$\frac{1}{2}\mu\delta^2$	E	$\frac{1}{2}\mu\delta^2$	$\log_{10}k'^2$	K	$\frac{1}{2}\mu\delta^2$	E	$\frac{1}{2}\mu\delta^2$
	3.		1.			3.		1.	
2.00	6956373630		0159935450		3.85	8666570181	8978	0119252774	22598
3.99	7070239428	11403	0156848495	29174	.84	8780737887	8824	0116931681	22186
.98	7184127841	11212	0153819360	28650	.83	8894923088	8672	0114654551	21781
.97	7298038489	11024	0150847007	28135	.82	9009125483	8523	0112420583	21384
.96	7411970999	10839	0147930414	27629	.81	9123344777	8377	0110228989	20993
3.95	7525925003	10656	0145068577	27131	3.80	9237580678	8232	0108078993	20609
.94	7639900138	10477	0142260510	26642	.79	9351832900	8090	0105969836	20232
.93	7753896048	10300	0139505242	26161	.78	9466101161	7950	0103900767	19861
.92	7867912382	10125	0136801819	25688	.77	9580385182	7812	0101871054	19497
.91	7981948794	9954	0134149305	25224	.76	9694684691	7677	0099879972	19139
3.90	8096004943	9785	0131546777	24767	3.75	9808999419	7543	0097926814	18787
.89	8210080493	9618	0128993329	24318	.74	9923329100	7412	0096010881	18442
.88	8324175115	9454	0126488073	23877	.73	*0037673475	7283	0094131488	18102
.87	8438288483	9293	0124030133	23443	.72	*0152032288	7156	0092287964	17768
.86	8552420277	9134	0121618648	23017	.71	*0266405285	7031	0090479645	17441

$\log_{10}c^{-2}$	K	$\frac{1}{2}\mu\delta^2$	E	$\frac{1}{2}\mu\delta^2$	$\log_{10}c^{-2}$	K	$\frac{1}{2}\mu\delta^2$	E	$\frac{1}{2}\mu\delta^2$
	4.		1.			4.		1.	
3.70	0380792220	6908	0088705883	17119	3.30	4965293552	3347	0039878708	8021
.69	0495192847	6786	0086966039	16802	.29	5080081337	3285	0039082714	7868
.68	0609606927	6667	0085259487	16492	.28	5194875631	3225	0038302304	7718
.67	0724034225	6550	0083585610	16186	.27	5309676315	3166	0037537181	7570
.66	0838474506	6435	0081943803	15887	.26	5424483272	3107	0036787053	7426
3.65	0952927543	6322	0080333472	15592	3.25	5539296386	3050	0036051634	7284
.64	1067393110	6210	0078754033	15303	.24	5654115544	2994	0035330641	7144
.63	1181870988	6100	0077204913	15018	.23	5768940633	2939	0034623799	7008
.62	1296360957	5992	0075685547	14739	.22	5883771545	2884	0033930837	6874
.61	1410862804	5886	0074195383	14465	.21	5998608172	2831	0033251490	6742
3.60	1525376318	5782	0072733877	14196	3.20	6113450407	2778	0032585495	6612
.59	1639901293	5679	0071300496	13931	.19	6228298148	2727	0031932597	6486
.58	1754437524	5578	0069894714	13671	.18	6343151291	2676	0031292545	6361
.57	1868984812	5479	0068516017	13416	.17	6458009737	2627	0030665092	6239
.56	1983542960	5381	0067163898	13165	.16	6572873387	2578	0030049995	6119
3.55	2098111773	5285	0065837862	12919	3.15	6687742145	2530	0029447017	6001
.54	2212691062	5191	0064537420	12677	.14	6802615915	2483	0028855924	5886
.53	2327280639	5098	0063262093	12440	.13	6917494603	2437	0028276489	5772
.52	2441880320	5007	0062011411	12207	.12	7032378120	2391	0027708486	5661
.51	2556489924	4917	0060784911	11978	.11	7147266373	2346	0027151695	5552
3.50	2671109273	4828	0059582139	11753	3.10	7262159275	2303	0026605899	5445
.49	2785738192	4742	0058402651	11532	.09	7377056738	2260	0026070886	5340
.48	2900376509	4657	0057246007	11315	.08	7491958679	2217	0025546448	5236
.47	3015024054	4573	0056111779	11102	.07	7606865011	2176	0025032381	5135
.46	3129680662	4490	0054999545	10893	.06	7721775655	2135	0024528484	5036
3.45	3244346168	4409	0053908890	10688	3.05	7836690528	2095	0024034561	4938
.44	3359020413	4330	0052839407	10486	.04	7951609550	2056	0023550417	4843
.43	3473703237	4251	0051790698	10288	.03	8066532646	2017	0023075864	4749
.42	3588394487	4174	0050762369	10094	.02	8181459736	1979	0022610716	4657
.41	3703094008	4098	0049754036	9903	.01	8296390748	1942	0022154790	4566
3.40	3817801651	4024	0048765320	9716	3.00	8411325606	1905	0021707908	4478
.39	3932517269	3951	0047795850	9532	4.99	8526264238	1869	0021269894	4391
.38	4047240715	3879	0046845262	9351	.98	8641206573	1834	0020840575	4306
.37	4161971849	3808	0045913198	9174	.97	8756152541	1799	0020419783	4222
.36	4276710530	3739	0044999307	9000	.96	8871102073	1765	0020007352	4140
3.35	4391456620	3671	0044103244	8829	4.95	8986055103	1732	0019603119	4059
.34	4506209983	3604	0043224670	8662	.94	9101011564	1699	0019206924	3980
.33	4620970488	3538	0042363252	8497	.93	9215971391	1667	0018818611	3903
.32	4735738003	3473	0041518665	8335	.92	9330934519	1635	0018438026	3826
.31	4850512400	3409	0040690589	8176	.91	9445900888	1604	0018065018	3752

$\log_{10}k'^2$	K	$\frac{1}{4}\mu\delta^2$	E	$\frac{1}{4}\mu\delta^2$	$\log_{10}k'^2$	K	$\frac{1}{4}\mu\delta^2$	E	$\frac{1}{4}\mu\delta^2$
	5.		1.			5.		1.	
$\bar{4}.90$	*9560870435	1574	0017699441	3679	$\bar{4}.50$	4161674347	724	0007773404	1659
.89	*9675843099	1544	0017341148	3607	.49	4276733006	709	0007614233	1626
.88	*9790818822	1515	0016989997	3536	.48	4391793070	696	0007458280	1593
.87	*9905797546	1486	0016645849	3467	.47	4506854511	682	0007305483	1562
.86	0020779213	1458	0016308567	3399	.46	4621917303	669	0007155778	1531
$\bar{4}.85$	0135763767	1430	0015978016	3333	$\bar{4}.45$	4736981420	656	0007009103	1500
.84	0250751153	1403	0015654065	3268	.44	4852046834	643	0006865398	1470
.83	0365741318	1376	0015336585	3204	.43	4967113522	630	0006724604	1441
.82	0480734207	1350	0015025448	3141	.42	5082181459	618	0006586662	1412
.81	0595729770	1324	0014720531	3079	.41	5197250619	606	0006451516	1384
$\bar{4}.80$	0710727956	1299	0014421711	3019	$\bar{4}.40$	5312320979	594	0006319111	1356
.79	0825728713	1274	0014128869	2960	.39	5427392515	582	0006189390	1329
.78	0940731993	1249	0013841887	2901	.38	5542465205	571	0006062301	1303
.77	1055737747	1225	0013560650	2844	.37	5657539025	560	0005937791	1277
.76	1170745928	1202	0013285046	2789	.36	5772613954	549	0005815809	1251
$\bar{4}.75$	1285756490	1179	0013014964	2734	$\bar{4}.35$	5887689970	538	0005696303	1226
.74	1400769387	1156	0012750294	2680	.34	6002767051	528	0005579225	1202
.73	1515784574	1134	0012490931	2627	.33	6117845177	517	0005464525	1177
.72	1630802007	1112	0012236770	2575	.32	6232924326	507	0005352157	1154
.71	1745821642	1091	0011987709	2525	.31	6348004480	497	0005242073	1131
$\bar{4}.70$	1860843438	1070	0011743646	2475	$\bar{4}.30$	6463085618	487	0005134227	1108
.69	1975867354	1049	0011504484	2426	.29	6578167720	478	0005028575	1086
.68	2090893347	1029	0011270126	2378	.28	6693250769	468	0004925073	1064
.67	2205921378	1009	0011040477	2331	.27	6808334744	459	0004823678	1043
.66	2320951408	990	0010815444	2285	.26	6923419629	450	0004724347	1022
$\bar{4}.65$	2435983398	971	0010594936	2240	$\bar{4}.25$	7038505404	441	0004627038	1001
.64	2551017311	952	0010378863	2196	.24	7153592052	432	0004531712	981
.63	2666053108	933	0010167138	2152	.23	7268679557	424	0004438328	961
.62	2781090755	915	0009959675	2110	.22	7383767901	415	0004346848	942
.61	2896130214	898	0009756388	2068	.21	7498857068	407	0004257233	923
$\bar{4}.60$	3011171452	880	0009557197	2027	$\bar{4}.20$	7613947041	399	0004169445	905
.59	3126214432	863	0009362019	1987	.19	7729037805	391	0004083448	886
.58	3241259123	847	0009170776	1948	.18	7844129343	384	0003999206	869
.57	3356305490	830	0008983388	1909	.17	7959221641	376	0003916684	851
.56	3471353501	814	0008799780	1871	.16	8074314683	369	0003835846	834
$\bar{4}.55$	3586403124	798	0008619877	1834	$\bar{4}.15$	8189408456	361	0003756659	817
.54	3701454329	783	0008443604	1797	.14	8304502943	354	0003679089	801
.53	3816507083	768	0008270891	1762	.13	8419598131	347	0003603105	784
.52	3931561357	753	0008101665	1727	.12	8534694007	340	0003528674	769
.51	4046617121	738	0007935859	1692	.11	8649790556	333	0003455764	753

$\log_{10} k'^2$	K	$\frac{1}{2}\mu\delta^2$	E	$\frac{1}{2}\mu\delta^2$	$\log_{10} k'^2$	K	$\frac{1}{2}\mu\delta^2$	E	$\frac{1}{2}\mu\delta^2$
	5., 6.		1.			6.		1.	
4.10	8764887765	327	0003384345	738	5.70	3369191819	146	0001462703	325
.09	8879985621	320	0003314387	723	.69	3484307262	143	0001432226	318
.08	8995084112	314	0003245860	708	.68	3599422986	140	0001402380	312
.07	9110183224	308	0003178736	694	.67	3714538988	137	0001373150	305
.06	9225282945	302	0003112986	680	.66	3829655260	134	0001344524	299
4.05	9340383264	296	0003048583	666	5.65	3944771798	131	0001316490	293
.04	9455484168	290	0002985498	653	.64	4059888596	129	0001289035	287
.03	9570585645	284	0002923706	640	.63	4175005649	126	0001262148	281
.02	9685687684	278	0002863180	627	.62	4290122952	124	0001235817	275
.01	9800790275	273	0002803894	614	.61	4405240500	121	0001210032	270
4.00	9915893405	267	0002745824	602	5.60	4520358287	119	0001184779	264
3.99	0030997065	262	0002688945	589	.59	4635476309	116	0001160050	259
.98	0146101243	257	0002633233	577	.58	4750594562	114	0001135832	253
.97	0261205929	252	0002578664	566	.57	4865713040	112	0001112116	248
.96	0376311114	247	0002525215	554	.56	4980831739	109	0001088891	243
3.95	0491416787	242	0002472863	543	5.55	5095950655	107	0001066147	238
.94	0606522938	237	0002421586	532	.54	5211069783	105	0001043874	233
.93	0721629558	232	0002371362	521	.53	5326189118	103	0001022063	228
.92	0836736638	227	0002322170	511	.52	5441308657	101	0001000704	224
.91	0951844168	223	0002273988	500	.51	5556428396	99	0000979787	219
3.90	1066952139	218	0002226797	490	5.50	5671548330	97	0000959305	215
.89	1182060543	214	0002180576	480	.49	5786668455	95	939247	210
.88	1297169371	210	0002135306	470	.48	5901788768	93	919605	206
.87	1412278614	206	0002090967	461	.47	6016909265	91	900371	202
.86	1527388263	201	0002047540	451	.46	6132029942	89	881536	198
3.85	1642498312	197	0002005006	442	5.45	6247150795	87	863092	193
.84	1757608752	193	0001963348	433	.44	6362271821	86	845031	189
.83	1872719574	190	0001922548	424	.43	6477393016	84	827345	186
.82	1987830772	186	0001882588	416	.42	6592514377	82	810026	182
.81	2102942338	182	0001843451	407	.41	6707635901	80	793067	178
3.80	2218054264	178	0001805120	399	5.40	6822757583	79	776461	174
.79	2333166543	175	0001767579	391	.39	6937879422	77	760199	171
.78	2448279168	171	0001730812	383	.38	7053001413	76	744276	167
.77	2563392132	168	0001694803	375	.37	7168123554	74	728683	164
.76	2678505428	164	0001659536	367	.36	7283245842	73	713415	160
3.75	2793619050	161	0001624997	360	5.35	7398368273	71	698464	157
.74	2908732992	158	0001591170	353	.34	7513490845	70	683825	154
.73	3023847245	155	0001558042	345	.33	7628613554	68	669490	151
.72	3138961805	152	0001525597	338	.32	7743736399	67	655453	148
.71	3254076665	149	0001493822	331	.31	7858859375	65	641708	145

$\log_{10}k'^2$	K	$\frac{1}{10}\delta^2$	E	$\frac{1}{10}\delta^2$	$\log_{10}k'^2$	K	$\frac{1}{10}\delta^2$	E	$\frac{1}{10}\delta^2$
	6., 7.		1.			7.		1.	
$\bar{5}.30$	7973982482	64	0000628249	142	$\bar{6}.90$	2578987753	28	0000268400	61
.29	8089105715	63	615071	139	.89	2694114403	27	262737	60
.28	8204229072	62	602167	136	.88	2809241106	27	257193	59
.27	8319352551	60	589532	133	.87	2924367863	26	251765	58
.26	8434476150	59	577160	130	.86	3039494672	26	246451	56
$\bar{5}.25$	8549599865	58	565046	128	$\bar{6}.85$	3154621531	25	241249	55
.24	8664723695	57	553184	125	.84	3269748441	25	236156	54
.23	8779847637	55	541570	122	.83	3384875399	24	231169	53
.22	8894971688	54	530197	120	.82	3500002405	24	226288	52
.21	9010095847	53	519062	117	.81	3615129458	23	221508	51
$\bar{5}.20$	9125220112	52	508159	115	$\bar{6}.80$	3730256557	23	216829	50
.19	9240344480	51	497483	113	.79	3845383701	22	212249	49
.18	9355468949	50	487031	110	.78	3960510889	22	207764	48
.17	9470593517	49	476796	108	.77	4075638119	21	203374	47
.16	9585718181	48	466775	106	.76	4190765393	21	199076	46
$\bar{5}.15$	9700842941	47	456963	104	$\bar{6}.75$	4305892707	20	194868	45
.14	9815967794	46	447355	101	.74	4421020062	20	190749	44
.13	9931092738	45	437949	99	.73	4536147457	20	186716	43
.12	0046217772	44	428739	97	.72	4651274890	19	182768	42
.11	0161342893	43	419721	95	.71	4766402361	19	178903	41
$\bar{5}.10$	0276468099	42	410892	93	$\bar{6}.70$	4881529870	18	175119	40
.09	0391593390	42	402247	91	.69	4996657415	18	171414	39
.08	0506718762	41	393783	89	.68	5111784996	18	167788	39
.07	0621844215	40	385495	88	.67	5226912612	17	164238	38
.06	0736969747	39	377381	86	.66	5342040262	17	160763	37
$\bar{5}.05$	0852095356	38	369437	84	$\bar{6}.65$	5457167945	17	157360	36
.04	0967221041	37	361659	82	.64	5572295662	16	154030	35
.03	1082346800	37	354043	81	.63	5687423410	16	150769	35
.02	1197472631	36	346587	79	.62	5802551190	16	147577	34
.01	1312598534	35	339286	77	.61	5917679001	15	144452	33
$\bar{5}.00$	1427724506	34	332139	76	$\bar{6}.60$	6032806842	15	141393	33
$\bar{6}.99$	1542850546	34	325141	74	.59	6147934713	15	138399	32
.98	1657976653	33	318290	72	.58	6263062613	14	135467	31
.97	1773102825	32	311582	71	.57	6378190541	14	132598	31
.96	1888229062	32	305014	70	.56	6493318496	14	129788	30
$\bar{6}.95$	2003355361	31	298584	68	$\bar{6}.55$	6608446479	13	127038	29
.94	2118481722	30	292289	67	.54	6723574489	13	124346	29
.93	2233608142	30	286126	65	.53	6838702524	13	121711	28
.92	2348734622	29	280091	64	.52	6953830586	13	119131	27
.91	2463861160	29	274184	63	.51	7068958672	12	116605	27

$\log_{10}k'^2$	K	$\frac{1}{2}\mu\delta^2$	E	$\frac{1}{2}\mu\delta^2$	$\log_{10}k'^2$	K	$\frac{1}{2}\mu\delta^2$	E	$\frac{1}{2}\mu\delta^2$
	7.		1.			8.		1.	
$\bar{0}.50$	7184086782	12	0000114133	26	$\bar{0}.25$	0062296182	7	0000066741	16
.49	7299214917	12	111713	26	.24	0177424778	7	65322	15
.48	7414343075	12	109344	25	.23	0292553388	7	63933	15
.47	7529471256	11	107025	25	.22	0407682011	7	62573	15
.46	7644599459	11	104755	24	.21	0522810647	7	61242	14
$\bar{0}.45$	7759727685	11	102532	24	$\bar{0}.20$	0637939296	6	59939	14
.44	7874855932	11	100357	23	.19	0753067959	6	58664	14
.43	7989984200	10	98228	23	.18	0868196633	6	57416	13
.42	8105112489	10	96143	22	.17	0983325320	6	56194	13
.41	8220240799	10	94103	22	.16	1098454019	6	54998	13
$\bar{0}.40$	8335369128	10	92105	21	$\bar{0}.15$	1213582729	6	53827	13
.39	8450497476	10	90150	21	.14	1328711451	6	52681	12
.38	8565625844	9	88236	20	.13	1443840184	6	51560	12
.37	8680754230	9	86362	20	.12	1558968929	5	50462	12
.36	8795882634	9	84528	20	.11	1674097683	5	49388	12
$\bar{0}.35$	8911011056	9	82733	19	$\bar{0}.10$	1789226449	5	48336	11
.34	9026139496	9	80976	19	.09	1904355225	5	47307	11
.33	9141267953	8	79256	18	.08	2019484010	5	46299	11
.32	9256396427	8	77572	18	.07	2134612806	5	45313	11
.31	9371524917	8	75924	18	.06	2249741612	5	44347	10
$\bar{0}.30$	9486653423	8	74310	17	$\bar{0}.05$	2364870426	5	43402	10
.29	9601781944	8	72731	17	.04	2479999251	5	42478	10
.28	9716910482	8	71185	17	.03	2595128084	4	41572	10
.27	9832039034	7	69672	16	.02	2710256926	4	40686	9
.26	9947167601	7	68191	16	.01	2825385777	4	39819	9
					$\bar{0}.00$	2940514636	4	38970	9

A NOTE ON BESSEL FUNCTIONS OF PURELY IMAGINARY ARGUMENT

By Elliott W. Montroll

1. Introduction

A difference-differential equation occurring frequently in physical problems is

$$ay_{n+1} - (a + b + d/dt)\, y_n + by_{n-1} = 0 \tag{1}$$

$$n = 1, 2, \cdots, N,$$

where $y_n(t)$ is a function of the discrete variable n and the continuous variable t, and a and b are constants. When N similar elements are arranged linearly and each element interacts with its neighbors (1) sometimes describes the transmission of various physical quantities through the elements. If t is interpreted to represent time, the transients in such systems can often be studied by application of the boundary conditions

$$y_1(t) = y_N(t) = 0 \text{ for all } t \tag{2a}$$

and the initial conditions

$$y_n(0) = \begin{cases} 1 \text{ if } n = m \\ 0 \text{ if } n \neq m. \end{cases} \tag{2b}$$

Under these conditions the solution of (1) is (as can be verified by direct substitution into (1) and (2))

$$y_n(t) = (b/a)^{(n-m)/2} e^{-(a+b)t} \sum_{k=-\infty}^{\infty} \cdot \{I_{n-m+2k(N-1)}(2t\sqrt{ab}) - I_{n+m-2+2k(N-1)}(2t\sqrt{ab})\} \tag{3}$$

where $I_n(x)$ is the n-th Bessel function of purely imaginary argument (here x is a real number) defined by the series

$$I_n(x) \equiv i^{-n} J_n(ix) \equiv \sum_{k=0}^{\infty} (x/2)^{n+2k}/k!\,(n+k)! \tag{4}$$

and which satisfies the differential equation

$$x^2 I_n''(x) + x\, I_n'(x) - (x^2 + n^2) I_n(x) = 0. \tag{5}$$

In this paper we shall develop some properties of $I_n(x)$ which are useful in examining the solution (3) of (1). More specifically, we shall locate the value of x which maximizes $e^{-\alpha x} I_n(x)$, find asymptotic formulae for

$$I_{n-1}(\beta n)/I_n(\beta n) \quad \text{and} \quad I_{n+1}(\beta n)/I_n(\beta n),$$

and derive an asymptotic expression for $I_n(\beta n)$ when n is large. These results will be used in a brief discussion of (3).

2. Location of the value of x which makes $f_n(\alpha, x) \equiv e^{-\alpha x} I_n(x)$ a maximum

From its definition $I_n(0) = 0$ when $n > 0$, and from the asymptotic expansion[1] (valid for large x)

$$I_n(x) \sim \frac{e^x}{(2\pi x)^{1/2}} \left\{ 1 + \sum_{r=1}^{\infty} (-1)^r \frac{[4n^2 - 1][4n^2 - 3^2] \cdots [4n^2 - (2r-1)^2]}{r!\, 2^{3r} x^r} \right\} \tag{5}$$

it is clear that when $\alpha \geq 1$, as $x \to \infty$, $e^{-\alpha x} I_n(x) \to 0$, and therefore that the continuous function $f_n(\alpha, x)$ has at least one maximum. Actually $f_n(\alpha, x)$ has only one maximum when $0 < x < \infty$ as we shall show later.

In this section we shall locate the value of x which makes $f_n(\alpha, x)$ a maximum by application of (4) and the well known recursion formulae[1] for $I_n(x)$:

$$I_{n-1}(x) - I_{n+1}(x) = (2n/x) I_n(x) \tag{6}$$

$$I_{n+1}(x) + I_{n-1}(x) = 2I'_n(x). \tag{7}$$

Obviously, if x_n is a point at which $f_n(\alpha, x)$ has a maximum:

$$df_n/dx = 0 \quad \text{and} \quad d^2 f_n/dx^2 < 0. \tag{8}$$

Therefore

$$I'_n(x_n) - \alpha I_n(x_n) = 0 \tag{9a}$$

and

$$I''_n(x_n) - 2\alpha I'_n(x_n) + \alpha^2 I_n(x_n) < 0. \tag{9b}$$

Combining (9a) with (7) we have

$$I_{n+1}(x_n) + I_{n-1}(x_n) = 2\alpha I_n(x_n) \tag{10}$$

and (9a) with (9b) gives us

$$I''_n(x_n) - \alpha^2 I_n(x_n) < 0. \tag{11}$$

Also (9a) and (5) yield

$$I''_n(x_n) = (1 + n^2/x_n^2 - \alpha/x_n) I_n(x_n). \tag{12}$$

The above relations and two lemmas to be proved presently will suffice to prove the main theorem of this section:

THEOREM I. If $\alpha > 1$, and x_n is the value of x which maximizes $f_n(\alpha, x) \equiv I_n(x) e^{-\alpha x}$, then

$$\alpha^2 - 1 + (1/x_n)(\alpha - n/x_n) < n^2/x_n^2 < \alpha^2 - 1 + \alpha/x_n, \tag{13}$$

$$\lim_{n \to \infty} x_n/n = 1/\sqrt{\alpha^2 + 1}, \tag{14a}$$

$$\text{and } x_n \sim n/\sqrt{\alpha^2 - 1}. \tag{14b}$$

The right hand half of the inequality (13) follows directly from (11) and (12):

$$n^2/x_n^2 < \alpha^2 - 1 + \alpha/x_n. \tag{13a}$$

[1] E. I. Whittaker and G. N. Watson, Modern Analysis, Chap. XVII, Cambridge, 1935.

To proceed further it is convenient to introduce two sequences

$$a_n \equiv I_{n-1}(x_n)/I_n(x_n) \quad \text{and} \quad b_n \equiv I_{n+1}(x_n)/I_n(x_n) \tag{15}$$

in terms of which (6) and (10) can be rewritten as

$$a_n = \alpha + n/x_n \quad \text{and} \quad b_n = \alpha - n/x_n\,. \tag{16}$$

Combination of (13a) and (16) yields

LEMMA 1. If a_n and b_n are defined by (15) then

a) $a_n < \alpha + \sqrt{\alpha^2 - 1 + \alpha/x_n}$

b) $b_n > \alpha - \sqrt{\alpha^2 - 1 + \alpha/x_n}\,.$

One other lemma is used to finally prove theorem I.

LEMMA 2. If x_n is the point at which $e^{-\alpha x}I_n(x)$ has its maximum value, then

$$a)\ I_n'(x_{n+1}) - \alpha I_n(x_{n+1}) < 0 \tag{17a}$$

$$b)\ I_n'(x_{n-1}) - \alpha I_n(x_{n-1}) > 0. \tag{17b}$$

To prove (a), let us assume that the contrary

$$I_n'(x_{n+1}) - \alpha I_n(x_{n+1}) \geq 0 \tag{18}$$

is true. Then eq. (7) implies

$$I_{n+1}(x_{n+1}) + I_{n-1}(x_{n+1}) - 2\alpha I_n(x_{n+1}) \geq 0.$$

Combining this with (6) and (16):

$$(1/a_n) + ((n-1)/x_n - \alpha) = (1/a_n)\left\{1 + \left(\frac{n-1}{x_n} - \alpha\right)\left(\alpha + \frac{n}{x_n}\right)\right\} \geq 0$$

or

$$1 - \alpha^2 + n^2/x_n^2 - (1/x_n)(\alpha + n/x_n) \geq 0.$$

However, from (13a) and the fact that $0 > -n/x_n^2$,

$$0 > 1 - \alpha^2 + (n/x_n)^2 - (1/x_n)(\alpha + n/x_n).$$

Thus our assumption (18) is false and part (a) of lemma 2 is proven.

To prove (b) we shall first prove that $f_n(\alpha, x)$ has only one maximum in the interval $0 \leq x < \infty$. If there are several maxima, there must be at least one minimum or an interval of zero slope between any two of the maxima. At a minimum point or a point in an interval of zero slope

$$I_n''(x) - \alpha^2 I_n(x) \geq 0 \quad \text{and} \quad I_n'(x) - \alpha I_n(x) = 0$$

have to be satisfied. Since a minimum or a point in an interval of zero slope between two maxima must occur at a value of x greater than the smallest x_n, (13a) implies

$$n^2 < x^2(\alpha^2 - 1 + \alpha/x_n).$$

This inequality combines with the differential equation (5) to yield

$$I_n''(x) = I_n(x)\left(1 + \frac{n^2 - \alpha x}{x^2}\right) < \alpha^2 I_n(x)$$

which is contrary to the condition necessary for the existence of a minimum or an interval of zero slope between two maxima. Thus there is only one value of x between zero and infinity which makes $f_n(\alpha, x)$ a maximum.

Since part (a) of our lemma is equivalent to $f_n'(\alpha, x_{n+1}) < 0$, the existence of a single maximum for $f_n(\alpha, x)$ implies $x_{n+1} > x_n$. Since this is true for all $n > 1$, $x_{n-1} < x_n$, which in turn implies that $f_n'(\alpha, x_{n-1}) > 0$. This inequality is equivalent to (17b), completing the proof of lemma 2.

We are now in a position to proceed with the proof of theorem I. (17b), (7), and (6) yield

$$I_{n-1}(x_{n-1}) - (\alpha + n/x_{n-1})I_n(x_{n-1}) > 0$$

which, upon introduction of the b_n's and (16) is equivalent to

$$0 < 1 - \alpha^2 + (n/x_n)^2 - (1/x_n)(\alpha - n/x_n).$$

This inequality plus (13a) gives us the first part of theorem I:

$$\alpha^2 - 1 + (1/x_n)(\alpha - n/x_n) < n^2/x_n^2 < \alpha^2 - 1 + \alpha/x_n.$$

Taking the limit of all parts of this inequality as $n \to \infty$,

$$\lim_{n\to\infty} x_n/n = \sqrt{\alpha^2 - 1},$$

proving the second part of theorem I.

As a biproduct of the relations used to prove theorem I, we can find some asymptotic properties of $\{a_n\}$ and $\{b_n\}$.

THEOREM II. If

$$a_n \equiv I_{n-1}(x_n)/I_n(x_n) \quad \text{and} \quad b_n \equiv I_{n+1}(x_n)/I_n(x_n),$$

then

a) $\alpha + \sqrt{\alpha^2 - 1 + (\alpha - n/x_n)/x_n} < a_n < \alpha + \sqrt{\alpha^2 - 1 + \alpha/x_n}$

b) $\alpha - \sqrt{\alpha^2 - 1 + (\alpha - n/x_n)/x_n} > b_n > \alpha - \sqrt{\alpha^2 - 1 + \alpha/x_n}$

and

c) $\lim a_n = \alpha + \sqrt{\alpha^2 - 1}$

d) $\lim b_n = \alpha - \sqrt{\alpha^2 - 1}.$

Parts a) and b) of this theorem are the obvious consequence of theorem I and eq. (16). Parts c) and d) of the theorem follow by taking limits of both sides of the inequalities in a) and b) as $n \to \infty$.

Furthermore, if we let

$$\beta = 1/\sqrt{\alpha^2 - 1}$$

and remember that

$$\lim_{n\to\infty} x_n/\beta n \to 1,$$

we have

THEOREM III. If $\beta > 0$,

a) $$\lim_{n\to\infty} I_{n-1}(\beta n)/I_n(\beta n) = (1/\beta)(\sqrt{\beta^2+1}+1)$$

b) $$\lim_{n\to\infty} I_{n+1}(\beta n)/I_n(\beta n) = (1/\beta)(\sqrt{\beta^2+1}-1).$$

When $\alpha = 1$, an analogue to theorem I can be found:

THEOREM IV. If x_n is the value of x which maximizes

$$f_n(1, x) = e^{-x}I_n(x),$$

then

$$x_n \sim n^2 + \tfrac{1}{2} + \cdots.$$

It is easy to show that

$$\lim_{n\to\infty} x_n/n^2 = 1 \quad \text{or} \quad x_n \sim n^2,$$

for according to (13a), when $\alpha = 1$, $x_n > n^2$, and from theorem I, if $n > 1$

$$x_n < n^2/(1 - n/x_n) < n^2/(1 - 1/n)$$

or

$$n^2 < x_n < n^2 + n + 2.$$

To obtain the next approximation to x_n one can use (5), which is an excellent representation of $I_n(x)$ when $x \simeq n^2$. By substituting (5) into (10) (when $\alpha = 1$) one can calculate the asymptotic root x_n of (10). Inasmuch as the calculation is somewhat long and tedious, we shall merely state that

$$x_n \sim n^2 + \tfrac{1}{2} + O(1/n)$$

and exhibit a table for the first five x_n's (as calculated numerically) which corroborates this result.

$n-$	1	2	3	4	5
x_n-	1·55	4·56	9·53	16·52	25·51

3. Asymptotic expression for $I_n(\beta n)$ for large integral n and $\beta > 0$

The analysis in this section has its root in the integral representation (1) of $I_n(u)$

$$I_n(u) = \frac{u^n}{2^n\Gamma(\frac{1}{2})\Gamma(n+\frac{1}{2})}\int_0^\pi \cosh(u\cos\theta)\sin^{2n}\theta\, d\theta. \tag{19}$$

When n is an integer this expression reduces to

$$I_n(\beta n) = (\beta n/2)^n G(\beta, n)/\Gamma(\tfrac{1}{2})\Gamma(n+\tfrac{1}{2}) \tag{20a}$$

where

$$G(\beta, n) = \int_0^{\pi} \exp n(\beta \cos\theta + 2 \log \sin\theta)\, d\theta. \tag{20b}$$

Since the coefficient of n in the integrand of (19) has a single maximum between 0 and π, the main contribution to the integral when n is large comes from values of θ in the neighborhood of this maximum. A method of evaluating integrals of this form is discussed in detail in the appendix and will be used here for the evaluation of $G(\beta, n)$.

From eq. 2 of the appendix, if $f(x)$ has the properties

$$\left.\begin{aligned} f'(\xi) &= 0 \\ f''(\xi) &= -2z < 0 \end{aligned}\right\} \text{ for some } a < \xi < b \tag{21a}$$

then

$$\int_a^b e^{nf(x)}\, dx \sim (\pi/nz)^{\frac{1}{2}} e^{nf(\xi)}(1 + \lambda_1/nz^3 + \lambda_2/(nz^3)^2 + \cdots) \tag{21b}$$

where

$$\lambda_1 = 1\cdot 3\left\{\frac{zf^{(4)}(\xi)}{2^2\cdot 4!} + \frac{5(f^{(3)}(\xi))^2}{2!\cdot 2^3(3!)^2}\right\} \tag{21c}$$

and λ_s's with higher subscripts are given by (A-12).

In the special case of eq. (20b)

$$f(\theta) = \beta \cos\theta + 2 \log \sin\theta,$$

and in the interval $\langle 0, \pi \rangle$, $f(\theta)$ has its maximum value at the point which is a root of

$$-\beta \sin^2\xi + 2\cos\xi = 0. \tag{22}$$

Also, at this point

$$f''(\xi) = -\beta\cos\xi - 2/\sin^2\xi = -2\sqrt{\beta^2 + 1} \tag{23a}$$

so

$$z = \sqrt{\beta^2 + 1} \tag{23b}$$

By calculating higher derivatives of $f(\theta)$ and applying (21c) it can be shown that

$$\lambda_1 = (-z^3 + 3z^2 - 5)/24 \tag{24a}$$

$$\lambda_2 = (z^6 - 6z^5 + 81z^4 + 10z^3 - 462z^2 + 385)/2(24)^2 \tag{24b}$$

$$\lambda_3 = (1003z^9 + 45z^8 - 1215z^7 + 30300z^6 + 6930z^5 - 369603z^4 - 5775z^3 + 765765z^2 - 425425)/30(24)^3. \tag{24c}$$

Introducing (A-13) into (20b)

$$G(\beta, n) \sim \sqrt{\pi/nz}\{2(z-1)/\beta^2\}^n \quad e^{n(z-1)}\{1 + \lambda_1/nz^3 + \cdots\}.$$

According to Stirling's formula for the Gamma Function

$$n^n/\Gamma(n+\tfrac{1}{2}) = (2\pi)^{-1/2}e^n\left(1+\frac{1}{24n}+\frac{1}{1152n^2}-\frac{1003}{414720n^3}+\cdots\right)$$

so, when $\beta > 0$

$$I_n(\beta n) \sim \frac{1}{\sqrt{2\pi nz}}\{e^z(z-1)/\beta\}^n\left\{1+\frac{(3\beta^2-2)}{24nz^3}+\frac{(81\beta^4-300\beta^2+4)}{2(24nz^3)^2}\right.$$
$$\left.+\frac{(30375\beta^6-278{,}478\beta^4+117{,}684\beta^2+1{,}112)}{30(24nz^3)^3}+\cdots\right\} \tag{25}$$

where

$$z = \sqrt{\beta^2+1}.$$

The first term in this formula was first obtained by Laplace[2] from the power series representation of $I_n(u)$. More recently Lehmer, using the differential equation for $I_n(u)$ has developed an asymptotic formula for $\log I_n(\beta n)$ which agrees with (25) to terms of $O(1/n^3)$. Lehmer[3] also calculates several higher order terms explicitly.

4. Discussion of the differential equation (1)

In this section we shall apply the results of sections 1–3 to the differential equation

$$ay_{n+1} - (a+b+d/dt)y_n + by_{n-1} = 0 \tag{1}$$

under the boundary and initial conditions (2). Moreover, we shall study (1) with a somewhat simplified set of conditions. Instead of letting n (which in this section will be called the index of an element) range from 0 to N, let

$$n = 0, \pm 1, \pm 2, \cdots, \pm\infty,$$

and suppose, for all t

$$y_{-\infty}(t) = y_{\infty}(t) = 0 \tag{26a}$$

and

$$y_n(0) = \begin{cases} 1 \text{ if } n = 0 \\ 0 \text{ if } n \neq 0. \end{cases} \tag{26b}$$

This alteration of the range of n is equivalent to the receding of the boundaries of (2) to $\pm\infty$, so the solution of (1) with these conditions should be the same as that under the old conditions (2) for values of n far from the boundary values 0 and N.

The solution of (1) in this case (26) involves only one Bessel function

$$y_n(t) = (b/a)^{n/2}e^{-(a+b)t}I_n(2t\sqrt{ab}) \tag{27}$$

[2] Laplace, Oeuvres, Tome 5, 489, Paris (1882).
[3] Lehmer, D. H., Mathematic Tables and Aids to Computation I, 133, (1944).

as we can see by introducing the generating function

$$G(t, z) = \sum_{n=-\infty}^{\infty} z^n (a/b)^{n/2} y_n(t), \tag{28}$$

and applying (1), yielding

$$dG/dt = (ab)^{\frac{1}{2}}(z - (a + b)/(ab)^{\frac{1}{2}} + 1/z)G, \tag{29}$$

and from the boundary conditions on $y_n(t)$

$$G(z, 0) = 1.$$

Solving (29) for $G(t, z)$:

$$G(t, z) = e^{-(a+b)t} \exp\{t(ab)^{\frac{1}{2}}(z + 1/z)\}. \tag{30}$$

Since[1] the n-th Bessel function of imaginary argument is given by

$$I_n(x) = \text{coef. of } z^n \text{ of } \exp\{\tfrac{1}{2} x(z + 1/z)\}$$

and since by definition $(ab)^{\frac{1}{2}n} y_n(t)$ is the coefficient of z^n in $G(t, z)$, $y_n(t)$ is given by (27). Also, from the generating function, for all t

$$\bar{n} = \sum_{n=-\infty}^{\infty} n y_n(t) = z\partial G(t, x)/\partial t]_{z=(b/a)^{\frac{1}{2}}} = t(b - a), \tag{31a}$$

$$\overline{n^2} = t(b + a) + t^2(b - a)^2, \tag{31b}$$

$$\sigma^2 = (\overline{n^2} - \bar{n}^2) = t(b + c). \tag{31c}$$

From (31a), when $b > a$, the variation of $y_n(t)$ from zero "travels" upward toward elements of higher n, while when $a > b$ the variation travels downward, and when $a = b$, $\bar{n}$ remains at $n = 0$.

The standard Bessel function tables can be used to obtain numerical results for (27) as long as $|\, n \,|$ is not too large (say <10). (27) is plotted in fig. 1 for several values of t when $a = 1$ and $b = 4$. The main features of this figure are

a) $y_n(t)$ has a single maximum, and with increasing t this maximum travels in the direction of increasing n,

b) the deviation of $y_n(t)$ spreads over more elements as t increases.

From these curves calculated for small t, one would expect each element to have a maximum deviation from 0 at some value of t. At the n-th element this would occur when

$$(d/dt)y_n(t) = 0. \tag{32}$$

On application of Theorem I, when n is large the value of t for which this is true is

$$t_n \sim n/(b - a). \tag{33}$$

Since $I_n(x) = I_{-n}(x)$, this value of t is the same for positive and negative n. Of course the value of $y_n(t)$ corresponding to t_n differs for $+n$ and $-n$ because of the factor $(b/a)^{n/2}$. It is also to be noted from (31a) that when $t = t_n$,

$$n \sim \bar{n} \tag{34}$$

and from (27), when n has the same sign as $\bar{n}$, $y_n(t_n) > y_{-n}(t_n)$.

We shall now show that when $t_n \sim n/(b-a)$, for sufficiently large n, $y_m(t_n)$ (as a function of m) has a maximum at $m = n$, i.e.

$$y_n(t_n) > \begin{cases} y_{n-1}(t_n) \\ y_{n+1}(t_n). \end{cases} \tag{35}$$

If $t > 0$, n must have the same sign as $n/(b-a)$. Then, by application of Theorem II

$$b_n \equiv \frac{I_{n+1}(2t_n\sqrt{ab})}{I_n(2t_n\sqrt{ab})} < \frac{a+b}{2\sqrt{ab}} - \sqrt{\frac{(b-a)^2}{4ab} + \left(\frac{a+b}{2\sqrt{ab}} - \frac{n}{x_n}\right)\frac{1}{x_n}}.$$

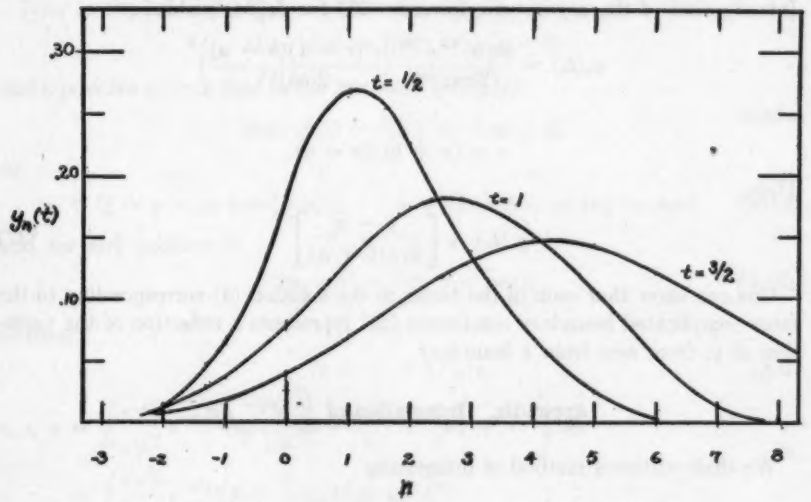

FIG. 1. VARIATION OF $y_n(t)$ WITH n FOR VARIOUS VALUES OF t

Since $n/x_n \sim (b-a)/2\sqrt{ab}$,

$$\frac{a+b}{2\sqrt{ab}} - \frac{n}{x_n} \sim \sqrt{a/b} > 0$$

and

$$b_n < \frac{a+b}{2\sqrt{ab}} - \frac{b-a}{2\sqrt{ab}} = \sqrt{a/b}$$

so that

$$y_{n+1}(t_n)/y_n(t_n) = (b/a)^{\frac{1}{2}}\, I_{n+1}(2t_n\sqrt{ab})/I_n(2t_n\sqrt{ab}) < 1$$

yielding $y_{n+1}(y_n) < y_n(t_n)$. Similarly, from part (a) of Theorem II $y_{n-1}(t_n) < y_n(t_n)$ proving (35). As n increases, it is also clear from Theorem II that $b_n \to (a/b)^{\frac{1}{2}}$ and $a_n \to (b/a)^{\frac{1}{2}}$ so that

$$y_{n+1}(t_n) \to y_n(t_n) \quad \text{and} \quad y_{n-1}(t_n) \to y_n(t_n) \tag{36}$$

and the maximum of $y_m(t_n)$ becomes less and less pronounced.

From (32) and (35) one can see that as long as $a \neq b$, $y_n(t)$ has its greatest variation from zero at $t = t_n$, and at this value of t $y_m(t)$ deviates more from zero at the n-th element than at neighboring elements. Thus, if t is interpreted as time in seconds, the initial variation of $y_0(t)$ from zero travels through other elements with a velocity

$$t_n/n \sim 1/(b - a) \text{ elements per second.}$$

At $t = t_n$ the magnitude of $y_n(t_n)$ is

$$y_n(t_n) \sim (b/a)^{\frac{1}{2}n} \exp\{-(a + b)n/(b - a)\} I_n(2n[ab]^{\frac{1}{2}}/[b - a]).$$

Introduction of the asymptotic formula (25) for $I_n(\beta n)$ yields

$$y_n(t_n) \sim \frac{(b/a)^{n/2} e^{-ns}}{(2\pi nz)^{1/2}} \left\{\frac{e^s(z - 1)(b - a)}{2(ab)^{1/2}}\right\}^n$$

where

$$z = (a + b)/(b - a).$$

Thus,

$$y_n(t_n) \sim \left[\frac{b - a}{2\pi n(b + a)}\right]^{1/2}. \tag{37}$$

One can show that each of the terms in the solution (3) corresponding to the more complicated boundary conditions (2a) represents a reflection of the variation of y_n from zero from a boundary.

Appendix. Integration of $\int_a^b e^{nf(x)}\,dx$

We shall outline a method of integrating

$$F(n) = \int_a^b e^{nf(x)}\,dx \tag{A-1}$$

when $f(x)$ is a continuous function having a single maximum in the interval $\langle a, b\rangle$ and n is large. The result is originally due to Laplace.[4] Suppose for some ξ such that $a < \xi < b$

$$f'(\xi) = 0, \qquad f''(\xi) = -2z < 0 \tag{A-2}$$

and

$$f(x) = f(\xi) + \sum_{k=2}^{\infty} (x - \xi)^k f^{(k)}(\xi)/k!. \tag{A-3}$$

Letting

$$a_k = f^{(k)}(\xi)/k!, \tag{A-4}$$

$$F(n) = e^{nf(\xi)}(s_1 + s_2 + s_3) \tag{A-5}$$

[4] Laplace, Oeuvres, Tome 7, 103, Paris (1882).

where

$$s_1 = \int_{\xi-\mu}^{\xi+\mu} \exp n\left\{\sum_{k=2}^{\infty} a_k (x-\xi)^k\right\} dx \tag{A-5a}$$

$$s_2 = \int_{a}^{\xi-\mu} \exp n\{f(x) - f(\xi)\}\, dx \tag{A-5b}$$

$$s_3 = \int_{\xi+\mu}^{b} \exp n\{f(x) - f(\xi)\}\, dx \tag{A-5c}$$

and μ is a constant such that $0 < \mu < \xi - a$.

Now in the interval $\langle a, \xi - \mu\rangle$ there is some positive α_1 such that

$$\text{Max } \{f(x) - f(\xi)\} = -\alpha_1 < 0$$

and a positive α_2 such that in the interval $\langle \xi + \mu, b\rangle$

$$\text{Max } \{f(x) - f(\xi)\} = -\alpha_2 < 0,$$

so

$$s_2 < (\xi - \mu - a)\exp(-\alpha_1 n), \qquad s_3 < (b - \xi - \mu)\exp(-\alpha_2 n) \tag{A-6a}$$

and for any positive l:

$$\lim_{n\to\infty} s_2 n^l = \lim_{n\to\infty} s_3 n^l = 0. \tag{A-6b}$$

Letting

$$(x - \xi) = h/\sqrt{n} \tag{A-7}$$

$$\begin{aligned} s_1\sqrt{n} &= \int_{-\mu\sqrt{n}}^{\mu\sqrt{n}} e^{-ah^2} \exp\left\{\frac{a_3 h^3}{\sqrt{n}} + \frac{a_4 h^4}{n} + \frac{a_5 h^5}{n^{3/2}} + \cdots\right\} dh \\ &= \int_{-\mu\sqrt{n}}^{\mu\sqrt{n}} e^{-ah^2}\left\{1 + \frac{1}{n}(a_4 h^4 + \tfrac{1}{2}a_3^2 h^4)\right. \\ &\quad \left. + \frac{1}{n^2}\left(a_6 h^6 + (2a_3 a_5 + a_4^2)\frac{h^8}{2!} + 3a_3^2 a_4 \frac{h^{10}}{3!} + a_3^4 \frac{h^{12}}{4!}\right) + \cdots\right\} dh. \end{aligned} \tag{A-8}$$

The coefficient of $(1/n)^k$ is

$$\mu_k = h^{2k} \sum_{m=1}^{2k} (h^{3m}/m!) \sum m! \prod_{i=1}^{m} a_{m_i}^{l_i}/l_i! \tag{A-9}$$

$$\left\{\begin{array}{c} \sum_i l_i m_i = 2(m+k) \\ m_i \geq 3 \end{array}\right\}$$

The factor on the right is to be interpreted as representing the sum of all products of terms

$$a_{m_i}^{l_i}/l_i!$$

for which

$$\sum l_i m_i = 2(m+k); \qquad l \leq i \leq m \text{ and } m \geq 3.$$

Now, by the mean value theorem, for sufficiently large n

$$\int_{\mu\sqrt{n}}^{\mu\sqrt{n}} e^{-sh^2}\,dh = \sqrt{\pi/z} - 2\int_{u\sqrt{n}}^{\infty} h^2 e^{-sh^2}\,(dh/h^2)$$
$$= \sqrt{\pi/z} + 2\eta\mu\sqrt{n}\,e^{-s\eta\mu^2}$$

where $|\eta| < 1$. Similarly

$$\int_{\mu\sqrt{n}}^{\mu\sqrt{n}} h^{2k} e^{-sh^2}\,dh = 1\cdot3\cdot5\cdots(2k-1)(1/2z)^k\sqrt{\pi/z}$$
$$- 2\int_{u\sqrt{n}}^{\infty} h^{2(k+1)} e^{-sh^2}\,(dh/h^2) \qquad \text{(A-10)}$$
$$= 1\cdot3\cdot5\cdots(2k-1)(1/2z)^k\sqrt{\pi/z} + 2\eta\mu^{2k+1}n^{k+\frac{1}{2}}e^{-s\eta\mu^2}.$$

Hence

$$\lim_{n\to\infty} n^j\{s_1\sqrt{n} - \sqrt{\pi/z}\,(1 + \lambda_1/nz^3 + \cdots + \lambda_j/(nz^3)^j\} = 0$$

and

$$\lim_{n\to\infty} n^j\{\sqrt{n}\,F(n)e^{-nf(\xi)} - \sqrt{\pi/z}\,(1 + \lambda_1/nz^3 + \cdots + \lambda_j/(nz^3)^j)\} = 0 \qquad \text{(A-11)}$$

where

$$\lambda_1 = 1\cdot3\left(\frac{za_4}{2^2} + \frac{5a_3^2}{2!\,2^3}\right),$$

$$\lambda_2 = 1\cdot3\cdot5\left\{\frac{z^3a_6}{2^3} + \frac{7z^2(2a_3a_5 + a_4^2)}{2!\,2^4} + \frac{7\cdot9z(3a_3^2a_4)}{3!\,2^5} + \frac{7\cdot9\cdot11a_3^4}{4!\,2^6}\right\}$$

$$\lambda_3 = 1\cdot3\cdot5\cdot7\left\{\frac{z^5a_8}{2^4} + \frac{9z^4(2a_3a_7 + 2a_4a_6 + a_5^2)}{2!\,2^5} + \frac{9\cdot11z^3(3a_3^2a_6 + 6a_3a_4a_5 + a_4^3)}{3!\,2^6}\right.$$
$$\left. + \frac{9\cdot11\cdot13z^2(4a_3^3a_5 + 6a_3^2a_4^2)}{4!\,2^7} + \frac{9\cdot11\cdot13\cdot15\cdot5za_3^4a_4}{5!\,2^8} + \frac{9\cdot11\cdot13\cdot15\cdot17a_3^6}{6!\,2^9}\right\}$$

$$\lambda_k = \sum_{j=1}^{2k} \frac{1\cdot3\cdots[2(k+j)-1]z^{(2k-j)}}{j!\,2^{j+k-1}} \sum j! \prod_{i=1}^{m} a_{m_i}^{l_i}/l_i\,! \qquad \text{(A-12)}$$
$$\left\{\sum l_i m_i = 2(m+k)\right\}$$

Yielding the asymptotic expressions for $F(n)$:

$$F(n) = \int_a^b e^{nf(x)}\,dx \sim \sqrt{\frac{\pi}{zn}}\,e^{nf(\xi)}(1 + \lambda_1/nz^3 + \lambda_2/(nz^3)^2 + \cdots). \qquad \text{(A-13)}$$

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LAGUERRE FUNCTIONS IN THE MATHEMATICAL FOUNDATIONS OF THE ELECTROMAGNETIC THEORY OF THE PARABOLOIDAL REFLECTOR

By Edmund Pinney

Introduction

With the development of modern microwave techniques, the use of the paraboloidal reflector as a device for concentrating and directing radiation has become of great importance. In the past a rigorous theory, of the paraboloidal reflector has failed of practical development because of the lack of knowledge of the properties of the mathematical functions in terms of which the paraboloidal wave functions are expressed. In the absence of such a theory, the physicists and engineers working with these devices have assumed radiation within the reflector to follow the laws of geometric optics.

This assumption is better, the greater the ratio of reflector diameter to wave length. However, even for diameters as great as ten or fifteen wave lengths, radiation patterns show considerable differences between experimental results and those predicted on the basis of the geometric optics theory.

The purpose of this paper is to establish a beginning point for the practical development of a more rigorous paraboloidal theory. This is done by developing in a systematic manner the properties of the functions in terms of which the paraboloidal wave functions are expressed. In this development are included not only many of the known properties of the Laguerre functions but a number of new ones whose investigation proved necessary to the development of the paraboloidal theory.

The scalar wave equation is separable in paraboloidal coordinates. The separated solution involves both solutions of Laguerre's differential equation, and it is necessary to study these solutions in some detail before paraboloidal problems can be solved.

First we develop the properties of $L_\nu^\mu(z)$ (here called Laguerre's *function*), which reduces to the Laguerre polynomial $L_n^\mu(x)$ whenever ν is a non-negative integer n. Laguerre polynomials have been very widely studied in the literature, and many of the results, included in §2 for completeness, have appeared in so many that it is difficult to determine their original authorship. This is not attempted, but the sources may be found in the bibliography, kindly provided by Professor Harry Bateman.

Other than trivial generalizations of Laguerre polynomial results to the more general Laguerre functions defined in this paper, there are a few new relationships which depend for their existence on the more general nature of the Laguerre functions. Also, certain of the integrals whose evaluation is required for the paraboloid theory seem to be new.

Next we develop the properties of the second solution, $U_\nu^\mu(z)$, of Laguerre's differential equation. This was defined in a somewhat different way and studied

to some extent by Bateman ("Electrical and Optical Wave Motion," Cambridge University Press) but seems otherwise to have received very little attention in the literature. Most of the results given here are therefore new.

In §4 certain relationships between the two solutions are given.

In §5 the auxiliary functions $S_\nu^\mu(z)$ and $V_\nu^\mu(z)$ are defined and briefly developed. Although quite simply related to the functions $L_\nu^\mu(z)$ and $U_\nu^\mu(z)$, they deserve special consideration because their use considerably simplifies the algebraic manipulations in the solution of paraboloidal problems.

In a subsequent paper, we plan to apply these results to the solution of certain of the more important problems of the paraboloidal reflector.

1. *The Wave Equation in Paraboloidal Coordinates*

From the theory of Maxwell's equations it follows that, in the absence of charge and current, the electric and magnetic vectors may be resolved into rectangular components, each of which satisfies the wave equation

$$\nabla^2\psi - \mu\epsilon\frac{\partial^2\psi}{\partial t^2} = 0, \tag{A}$$

where μ and ϵ are the magnetic and electric inductive capacities of the medium. If, as we shall henceforth assume, the time enters ψ in the factor $e^{-i\omega t}$, then

$$\nabla^2\psi + k^2\psi = 0, \tag{B}$$

where

$$k^2 = \omega^2\mu\epsilon. \tag{C}$$

We shall investigate solutions of (B) in paraboloidal coordinates. These are defined in terms of rectangular coordinates x, y, z by

$$x = 2\sqrt{\xi\eta}\cos\varphi, \qquad y = 2\sqrt{\xi\eta}\sin\varphi, \qquad z = -\xi + \eta. \tag{1.1}$$

Writing $\rho^2 = x^2 + y^2$,

$$\rho^2 = 4\xi\eta = 4\xi^2 + 4\xi z = 4\eta^2 - 4\eta z. \tag{1.2}$$

Assuming $\xi \geqq 0$, $\eta \geqq 0$ so that ρ is real, we see that $\xi = $ const. and $\eta = $ const. correspond to paraboloids of revolution about the z-axis, open in the positive and negative directions of the z-axis respectively. These families of paraboloids are confocal, with the origin as focus, and have focal distances ξ and η respectively.

Now

$$\begin{aligned} ds^2 &= d\rho^2 + \rho^2 d\varphi^2 + dz^2, \\ \rho d\rho &= 2\xi d\eta + 2\eta d\xi, \\ dz &= -d\xi + d\eta. \\ \rho^2(d\rho^2 + dz^2) &= (2\xi d\eta + 2\eta d\xi)^2 + 4\xi\eta(d\xi - d\eta)^2 \\ &= 4\eta(\xi + \eta)d\xi^2 + 4\xi(\xi + \eta)d\eta^2. \end{aligned}$$

$$\therefore d\rho^2 + dz^2 = \frac{\xi+\eta}{\xi}\,d\xi^2 + \frac{\xi+\eta}{\eta}\,d\eta^2.$$

Therefore

$$ds^2 = \frac{\xi+\eta}{\xi}\,d\xi^2 + \frac{\xi+\eta}{\eta}\,d\eta^2 + 4\xi\eta\,d\varphi^2. \tag{1.3}$$

Therefore

$$h_1 = \sqrt{\frac{\xi+\eta}{\xi}}, \qquad h_2 = \sqrt{\frac{\xi+\eta}{\eta}}, \qquad h_3 = 2\sqrt{\xi\eta}. \tag{1.4}$$

Now

$$\nabla^2\psi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial\xi}\left(\frac{h_2h_3}{h_1}\frac{\partial\psi}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(\frac{h_3h_1}{h_2}\frac{\partial\psi}{\partial\eta}\right) + \frac{\partial}{\partial\varphi}\left(\frac{h_1h_2}{h_3}\frac{\partial\psi}{\partial\varphi}\right)\right].$$

$$h_1h_2h_3 = 2(\xi+\eta)$$

$$\frac{h_2h_3}{h_1} = 2\xi, \qquad \frac{h_3h_1}{h_2} = 2\eta, \qquad \frac{h_1h_2}{h_3} = \frac{\xi+\eta}{2\xi\eta}.$$

Therefore

$$\nabla^2\psi = \frac{1}{\xi+\eta}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial\psi}{\partial\xi}\right) + \frac{1}{\xi+\eta}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial\psi}{\partial\eta}\right) + \frac{1}{4\xi\eta}\frac{\partial^2\psi}{\partial\varphi^2}. \tag{1.5}$$

From (B),

$$\frac{\partial}{\partial\xi}\left(\xi\frac{\partial\psi}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(\eta\frac{\partial\psi}{\partial\eta}\right) + \frac{\xi+\eta}{4\xi\eta}\frac{\partial^2\psi}{\partial\varphi^2} + k^2(\xi+\eta)\psi = 0. \tag{1.6}$$

If we write

$$\psi(\xi,\eta,\varphi) = \psi^\mu(\xi,\eta)e^{\pm i\mu\varphi}, \tag{1.7}$$

we have

$$\frac{\partial}{\partial\xi}\left(\xi\frac{\partial\psi^\mu}{\partial\xi}\right) + \left(-\frac{\mu^2}{4\xi} + k^2\xi\right)\psi^\mu + \frac{\partial}{\partial\eta}\left(\eta\frac{\partial\psi^\mu}{\partial\eta}\right) + \left(-\frac{\mu^2}{4\eta} + k^2\eta\right)\psi^\mu = 0.$$

This equation may be separated in the form

$$\psi^\mu(\xi,\eta) = \psi^\mu_\nu(\pm\xi)\psi^\mu_\nu(\mp\eta), \tag{1.8}$$

ν depending on the separation constant, h, in a manner which will be defined later. Then

$$\frac{1}{\psi^\mu_\nu(\pm\xi)}\frac{d}{d\xi}\left[\xi\frac{d}{d\xi}\psi^\mu_\nu(\pm\xi)\right] - \frac{\mu^2}{4\xi} + k^2\xi$$

$$+ \frac{1}{\psi^\mu_\nu(\mp\eta)}\frac{d}{d\eta}\left[\eta\frac{d}{d\eta}\psi^\mu_\nu(\mp\eta)\right] - \frac{\mu^2}{4\eta} + k^2\eta = 0.$$

Therefore,

$$\frac{d}{dx}\left[x\frac{d}{dx}\psi_\nu^\mu(x)\right]+\left(k^2x+h-\frac{\mu^2}{4x}\right)\psi_\nu^\mu(x)=0. \tag{1.9}$$

Now let

$$\psi_\nu^\mu(x)=x^{\mu/2}e^{-ikx}\phi_\nu^\mu(x).$$

$$x\frac{d\psi_\nu^\mu}{dx}=x^{\mu/2}e^{-ikx}\left[x\frac{d\phi_\nu^\mu}{dx}+\left(\frac{\mu}{2}-ikx\right)\phi_\nu^\mu\right]$$

$$\frac{d}{dx}\left[x\frac{d\psi_\nu^\mu}{dx}\right]=x^{\mu/2}e^{-ikx}\left[\frac{d}{dx}\left(x\frac{d\phi_\nu^\mu}{dx}\right)+(\mu-2ikx)\frac{d\phi^\mu}{dx}\right. \tag{1.10}$$

$$\left.+\left\{-ik+\left(\frac{\mu}{2}-ikx\right)^2\frac{1}{x}\right\}\phi_\nu^\mu\right].$$

From (1.9),

$$\frac{d}{dx}\left(x\frac{d\phi_\nu^\mu}{dx}\right)+(\mu-2ikx)\frac{d\phi_\nu^\mu}{dx}$$

$$+\left\{-ik+\frac{\mu^2}{4x}-\mu ik-k^2x+k^2x+h-\frac{\mu^2}{4x}\right\}\phi_\nu^\mu=0.$$

Therefore if we take

$$h=ik(\mu+2\nu+1),$$

we have

$$x\frac{d^2\phi_\nu^\mu}{dx^2}+(\mu+1-2ikx)\frac{d\phi_\nu^\mu}{dx}+2ik\nu\phi_\nu^\mu=0.$$

Therefore

$$\phi_\nu^\mu(x)=y_\nu^\mu(2ikx) \tag{1.11}$$

where y satisfies

$$x\frac{d^2y}{dx^2}+(\mu+1-x)\frac{dy}{dx}+\nu y=0. \tag{1.12}$$

This may be recognized as the equation satisfied by the Laguerre function, whose definition may be written

$$L_\nu^\mu(z)=-\frac{\sin\pi\nu}{\pi}\Gamma(1+\mu+\nu)\sum_0^\infty{}_p\frac{\Gamma(p-\nu)}{\Gamma(p+\mu+1)}\frac{z^p}{p!} \tag{1.13}$$

This definition is not entirely satisfactory because $L_\nu^\mu\equiv 0$ for ν a negative integer, and $L_\nu^\mu\equiv\infty$ for $\mu+\nu$ a negative integer, except when $\mu=-m$, $\nu=n$, m and n being integers such that $0\leqq n<m$. However finite, non-null solutions of (1.12) certainly exist in these cases. The trouble lies in the factor multiplying the series in (1.13), and one might remedy it by changing this factor. However it seems impossible to remedy it completely without seriously damaging the simplicity of the recurrence relations satisfied by the Laguerre

functions. In view of the fact that our applications rarely will take us into the regions of difficulty, and in view of the extensive existing literature employing the definition of (1.13), we shall use it. However it must be borne in mind that although L_ν^μ, considered as a function of μ and ν, may approach infinity or zero as the quantities μ and ν approach the values mentioned above, it may occur in algebraic expressions which become indeterminate for these values, giving rise to finite non-null expressions.

Equation (1.12) has a second solution which we will denote by $U_\mu^\nu(x)$, and define later. Therefore we see that we may write our separated solutions of (B) in the form

$$\begin{aligned}\psi = \xi^{\mu/2}\eta^{\mu/2}e^{\pm ik(\xi-\eta)}\{AL_\nu^\mu(\mp 2ik\xi) + BU_\nu^\mu(\mp 2ik\xi)\} \\ \times \{CL_\nu^\mu(\pm 2ik\eta) + DU_\nu^\mu(\pm 2ik\eta)\}\{Ee^{i\omega\varphi} + Fe^{-i\omega\varphi}\}e^{-i\omega t}\end{aligned} \tag{1.14}$$

2. *Laguerre Functions*

For brevity let us adopt the convention that, unless specifically noted otherwise, Latin small case letters will be used to denote integers, when used as indices.

We define

$$L_\nu^\mu(z) = -\frac{\sin \pi\nu}{\pi}\Gamma(\mu+\nu+1)\sum_0^\infty{}_p \frac{\Gamma(p-\nu)}{\Gamma(p+\mu+1)}\frac{z^p}{p!}, \tag{2.1}$$

which is an entire function of z. In particular

$$\begin{aligned}L_n^\mu(z) &= \sum_0^n{}_p \frac{\Gamma(n+\mu+1)}{\Gamma(p+\mu+1)(n-p)!}\frac{(-z)^p}{p!} \\ &= \sum_0^n{}_p \binom{n+\mu}{n-p}\frac{(-z)^p}{p!}, \qquad n \geq 0.\end{aligned} \tag{2.2}$$

From §14.11 of Whittaker and Watson,

$$\sum_0^n{}_p \binom{n}{p}(-)^p \frac{\Gamma(p+\mu+\nu+1)}{\Gamma(p+\mu+1)} = \frac{\Gamma(\mu+\nu+1)\Gamma(n-\nu)}{\Gamma(-\nu)\Gamma(n+\mu+1)}, \quad n \geq 0. \tag{2.3}$$

Now

$$\begin{aligned}e^z L_{-\mu-\nu-1}^\mu(-z) &= \frac{\sin \pi(\mu+\nu+1)}{\pi}\Gamma(-\nu)\sum_0^\infty{}_p\sum_0^\infty{}_q \frac{(-)^p}{p!q!}\frac{\Gamma(p+\mu+\nu+1)}{\Gamma(p+\mu+1)}z^{p+q} \\ &= \frac{\sin \pi(\mu+\nu+1)}{\pi}\Gamma(-\nu)\sum_0^\infty{}_n \frac{z^n}{n!}\sum_0^\infty{}_p \binom{n}{p}(-)^p \frac{\Gamma(p+\mu+\nu+1)}{\Gamma(p+\mu+1)} \\ &= \frac{\sin \pi(\mu+\nu+1)}{\pi}\Gamma(\mu+\nu+1)\sum_0^\infty{}_n \frac{\Gamma(n-\nu)}{\Gamma(n+\mu+1)}\frac{z^n}{n!} \\ &= \frac{\sin \pi(\mu+\nu)}{\sin \pi\nu}L_\nu^\mu(z)\end{aligned}$$

$$L_\nu^\mu(z) = \frac{\sin \pi\nu}{\sin \pi(\mu+\nu)}e^z L_{-\mu-\nu-1}^\mu(-z). \tag{2.4}$$

$$\sum_0^\infty{}_n \frac{\Gamma(n-\nu)}{\Gamma(n+\mu+1)}\, h^n L_n^\mu(z) = \sum_0^\infty{}_n \Gamma(n-\nu) h^n \sum_0^n{}_p \frac{(-z)^p}{p!\,(n-p)!\,\Gamma(p+\mu+1)}$$

$$= \sum_0^\infty{}_p \frac{(-hz)^p}{p!\,\Gamma(p+\mu+1)} \sum_0^\infty{}_q \Gamma(q+p-\nu)\frac{h^q}{q!}$$

$$= \sum_0^\infty{}_p \frac{\Gamma(p-\nu)}{\Gamma(p+\mu+1)} \frac{(-hz)^p}{p!}\,(1-h)^{\nu-p}$$

$$= -\frac{\pi}{\sin \pi\nu}\frac{1}{\Gamma(\mu+\nu+1)}(1-h)^\nu L_\nu^\mu\left(-\frac{hz}{1-h}\right).$$

Therefore

$$(1-h)^\nu L_\nu^\mu\left(-\frac{hz}{1-h}\right) \tag{2.5}$$

$$= -\frac{\sin \pi\nu}{\pi}\Gamma(\mu+\nu+1)\sum_0^\infty{}_n \frac{\Gamma(n-\nu)}{\Gamma(n+\mu+1)} h^n L_n^\mu(z), \qquad |h|<1.$$

From (2.4),

$$(1-h)^{-\mu-\nu-1} e^{-hz/(1-h)} L_\nu^\mu\left(\frac{hz}{1-h}\right) \tag{2.6}$$

$$= \frac{1}{\Gamma(\nu+1)}\sum_0^\infty{}_n \frac{\Gamma(n+\mu+\nu+1)}{\Gamma(n+\mu+1)} h^n L_n^\mu(z), \qquad |h|<1.$$

From (2.2), setting $\nu = 0$,

$$(1-h)^{-\mu-1} e^{-hz/(1-h)} = \sum_0^\infty{}_n h^n L_n^\mu(z), \qquad |h|<1. \tag{2.7}$$

From §16.1 of Whittaker and Watson,

$$L_\nu^\mu(z) = \frac{\Gamma(\mu+\nu+1)}{\Gamma(\mu+1)\Gamma(\nu+1)}\, z^{-\frac{1}{2}\mu-\frac{1}{4}} e^{\frac{1}{2}z} M_{\frac{1}{2}\mu+\nu+\frac{1}{4},\frac{1}{2}\mu}(z) \tag{2.8}$$

From §§16.11, 16.41 of W. and W.,

$$\left.\begin{aligned} L_\nu^\mu(z) &= z^{-\frac{1}{2}\mu-\frac{1}{4}} e^{\frac{1}{2}z}\left\{\frac{e^{-\pi i\nu}}{\Gamma(\nu+1)} W_{\frac{1}{2}\mu+\nu+\frac{1}{4},\frac{1}{2}\mu}(z)\right. \\ &\qquad \left. - \frac{\sin \pi\nu}{\pi}\, e^{-\pi i(\frac{1}{2}\mu+\nu+\frac{1}{4})}\Gamma(\mu+\nu+1) W_{-\frac{1}{2}\mu-\nu-\frac{1}{4},\frac{1}{2}\mu}(-z)\right\}, \\ &\qquad\qquad -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}, \\ L_\nu^\mu(z) &= z^{-\frac{1}{2}\mu-\frac{1}{4}} e^{\frac{1}{2}z}\left\{\frac{e^{\pi i\nu}}{\Gamma(\nu+1)} W_{\frac{1}{2}\mu+\nu+\frac{1}{4},\frac{1}{2}\mu}(z)\right. \\ &\qquad \left. - \frac{\sin \pi\nu}{\pi}\, e^{\pi i(\frac{1}{2}\mu+\nu+\frac{1}{4})}\Gamma(\mu+\nu+1) W_{-\frac{1}{2}\mu-\nu-\frac{1}{4},\frac{1}{2}\mu}(-z)\right\}, \\ &\qquad\qquad -\frac{3\pi}{2} < \arg z < \frac{\pi}{2}. \end{aligned}\right\} \tag{2.9}$$

From §16.3 of W. and W., we obtain the asymptotic expression,

$$\left.\begin{aligned} L_\nu^\mu(z) \sim & -\frac{\sin \pi\nu}{\pi\Gamma(\nu+1)} z^{-\mu-\nu-1} e^z \sum_0^\infty{}_n \Gamma(n+\nu+1)\Gamma(n+\mu+\nu+1)\frac{1}{n!z^n} \\ & + e^{\mp\pi i\nu}\Gamma(\mu+\nu+1) z^\nu \sum_0^\infty{}_n \\ & \times \frac{1}{\Gamma(-n+\nu+1)\Gamma(-n+\mu+\nu+1)} \frac{(-)^n}{n!z^n}, \end{aligned}\right\} \tag{2.10}$$

where the upper or lower sign is taken, according as $0 < \arg z < \pi$ or $-\pi < \arg z < 0$, respectively. Unless otherwise noted, we shall continue to use this convention throughout this section

For $|z| \gg 1$,

$$L_\nu^\mu(z) \sim -\frac{\sin \pi\nu}{\pi}\Gamma(\mu+\nu+1) z^{-\mu-\nu-1} e^z + \frac{e^{\mp\pi i\nu}}{\Gamma(\nu+1)} z^\nu. \tag{2.11}$$

In particular, for $|z| \gg 1$,

$$L_n^\mu(z) \sim \frac{(-)^m}{n!} z^n, \qquad n \geq 0. \tag{2.12}$$

When ν is not a non-negative integer, for $|z| \gg 1$,

$$\left.\begin{aligned} L_\nu^\mu(z) &\sim -\frac{\sin \pi\nu}{\pi}\Gamma(\mu+\nu+1) z^{-\mu-\nu-1} e^z, && -\frac{\pi}{2} < \arg z < \frac{\pi}{2}, \\ L_\nu^\mu(z) &\sim \frac{e^{-\pi i\nu}}{\Gamma(\nu+1)} z^\nu, && \frac{\pi}{2} < \arg z < \pi, \\ L_\nu^\mu(z) &\sim \frac{e^{\pi i\nu}}{\Gamma(\nu+1)} z^\nu, && -\pi < \arg z < -\frac{\pi}{2}. \end{aligned}\right\} \tag{2.13}$$

For $x \gg 1$,

$$\left.\begin{aligned} L_\nu^\mu(\pm ix) &\sim \frac{e^{\mp\frac{1}{2}\pi i\nu}}{\Gamma(\nu+1)} x^\nu, && R(\mu+2\nu) > -1, \\ L_\nu^\mu(\pm ix) &\sim -e^{\mp\frac{1}{2}\pi i(\mu+\nu+1)} \frac{\sin \pi\nu}{\pi}\Gamma(\mu+\nu+1) x^{-\mu-\nu-1} e^{\pm ix}, \\ & && R(\mu+2\nu) < -1, \\ L_\nu^\mu(\pm ix) &\sim -e^{\mp\frac{1}{2}\pi i(\mu+\nu+1)} \frac{\sin \pi\nu}{\pi}\Gamma(\mu+\nu+1) x^{-\mu-\nu-1} e^{\pm ix} + \frac{e^{\mp\frac{1}{2}\pi i\nu}}{\Gamma(\nu+1)} x^\nu, \\ & && R(\mu+2\nu) = -1, \end{aligned}\right\} \tag{2.14}$$

where the sign convention above does not apply here, of course.

From (2.1), using Stirling's formula, for $R(\nu) > 0$

$$\lim_{s\to 0}\left(\frac{s}{\nu}\right)^\mu L_{\nu/s}^\mu(sz) = \lim_{s\to 0}\left(\frac{s}{\nu}\right)^\mu \frac{\Gamma(\mu+\nu/s+1)}{\Gamma(\nu/s+1)}$$

$$\times \lim_{s\to 0} \sum_0^\infty{}_p \frac{(-\nu/s)(-\nu/s+1)\cdots(-\nu/s+p-1)}{\Gamma(p+\mu+1)} \frac{(sz)^p}{p!}$$

$$= \lim_{s\to 0} \sum_0^\infty{}_p \frac{\nu[\nu-s]\cdots[\nu-s(p-1)]}{\Gamma(p+\mu+1)} \frac{(-z)^p}{p!}$$

For νz and s bounded, this summation is uniformly convergent in νz and s, so the order of the limit and summation may be reversed, and

$$\lim_{s\to 0} \left(\frac{s}{\nu}\right)^\mu L^\mu_{\nu/s}(sz) = (\nu z)^{-\frac{1}{2}\mu} J_\mu(2\sqrt{\nu z}), \tag{2.15}$$

uniformly in νz if νz is bounded* and $R(\nu) > 0$.

Since (2.1) is uniformly convergent,

$$\int_0^\infty e^{-t}\, t^\lambda\, L^\mu_\nu(zt)\, dt = -\frac{\sin \pi\nu}{\pi}\, \Gamma(\mu+\nu+1) \sum_0^\infty{}_p \frac{\Gamma(p-\nu)}{\Gamma(p+\mu+1)} \frac{z^p}{p!} \int_0^\infty e^{-t}\, t^{\lambda+p}\, dt$$

$$= -\frac{\sin \pi\nu}{\pi}\, \Gamma(\mu+\nu+1) \sum_0^\infty{}_p \frac{\Gamma(p-\nu)\Gamma(p+\lambda+1)}{p!\,\Gamma(p+\mu+1)} z^p$$

$$= \frac{\Gamma(\lambda+1)\Gamma(\mu+\nu+1)}{\Gamma(\mu+1)\Gamma(\nu+1)} F(-\nu, \lambda+1; \mu+1; z).$$

This derivation is valid when $|z| < 1$ and $R(\lambda) > -1$. In view of (2.12), (2.13), (2.14), and §5.32 of W. and W., this relation may be extended analytically so that

$$\left.\begin{aligned} &\int_0^\infty e^{-t}\, t^\lambda\, L^\mu_\nu(zt)\, dt = \frac{\Gamma(\lambda+1)\Gamma(\mu+\nu+1)}{\Gamma(\mu+1)\Gamma(\nu+1)} F(-\nu, \lambda+1; \mu+1; z), \\ &\nu = n \geq 0, \qquad R(\lambda) > -1, \qquad \text{or} \quad R(z) < 1, \qquad R(\lambda) > -1, \\ &\text{or } R(z) = 1, \qquad R(\mu+\nu-\lambda) > 0, \qquad R(\lambda) > -1. \end{aligned}\right\} \tag{2.16}$$

In particular, for $m \geq 0$,

$$\int_0^\infty e^{-t}\, t^{\mu+m}\, L^\mu_\nu(zt)\, dt$$

$$= -\frac{\sin \pi\nu}{\pi}\, \Gamma(\mu+\nu+1) \sum_0^\infty{}_p \frac{\Gamma(p-\nu)\Gamma(p+\mu+m+1)}{p!\,\Gamma(p+\mu+1)} z^p$$

$$= -\frac{\sin \pi\nu}{\pi}\, \Gamma(\mu+\nu+1)\, z^{-\mu} \frac{d^m}{dz^m} \left\{ z^{m+\mu} \sum_0^\infty{}_p \Gamma(p-\nu) \frac{z^p}{p!} \right\}$$

$$= \frac{\Gamma(\mu+\nu+1)}{\Gamma(\nu+1)} z^{-\mu} \frac{d^m}{dz^m} \{ z^{m+\mu}(1-z)^\nu \}.$$

Extending analytically, as before,

* See Th. 8.1.3, G. Szegö, "Orthogonal polynomials" (Amer. Math. Soc. Colloquium Publications, vol. 23, New York, 1939).

$$\left.\begin{aligned}&\int_0^\infty e^{-t}\, t^{\mu+m}\, L_\nu^\mu(zt)\, dt = \frac{\Gamma(\mu+\nu+1)}{\Gamma(\nu+1)}\, z^{-\mu}\frac{d^m}{dz^m}\{z^{\mu+m}(1-z)^\nu\},\\ &\qquad \nu = n \geq 0, \quad m \geq 0, \quad R(\mu+m) > -1,\\ &\text{or}\quad R(z) < 1, \quad m \geq 0, \quad R(\mu+m) > -1,\\ &\text{or}\quad R(z) = 1, \quad R(\nu) > m \geq 0, \quad R(\mu+m) > -1.\end{aligned}\right\} \tag{2.17}$$

Applying Leibnitz's rule to this,

$$\begin{aligned}\int_0^\infty e^{-t}\, t^{\mu+m}\, L_\nu^\mu(zt)\, dt &= \frac{\Gamma(\mu+\nu+1)}{\Gamma(\nu+1)}\, z^{-\mu}\sum_0^m{}_p \binom{m}{p}\frac{d^{m-p}}{dz^{m-p}}\, z^{m+\mu}\frac{d^p}{dz^p}(1-z)^\nu\\ &= \frac{\Gamma(\mu+\nu+1)}{\Gamma(\nu+1)}\, z^{-\mu}\sum_0^m{}_p \binom{m}{p}\frac{\Gamma(m+\mu+1)}{\Gamma(p+\mu+1)}\\ &\qquad \times z^{p-\mu}\frac{\Gamma(\nu+1)}{\Gamma(\nu+1-p)}(-)^p(1-z)^{\nu-p}\\ &= -\Gamma(\mu+\nu+1)\Gamma(m+\mu+1)\frac{\sin \pi\nu}{\pi}(1-z)^\nu\\ &\qquad \times \sum_0^m{}_p \binom{m}{p}\frac{\Gamma(p-\nu)}{\Gamma(p+\mu+1)}\left(\frac{z}{1-z}\right)^p\end{aligned}$$

Therefore,

$$\left.\begin{aligned}&\int_0^\infty e^{-t}\, t^{\mu+m}\, L_n^\mu(zt)\, dt = \Gamma(\mu+n+1)\Gamma(\mu+m+1)(1-z)^n\\ &\qquad \times \sum_0^m{}_p \frac{\binom{m}{p}}{(n-p)!\,\Gamma(p+\mu+1)}\left(\frac{-z}{1-z}\right)^p,\\ &\qquad n, m \geq 0, \quad R(\mu+m) > -1,\\ &\int_0^\infty e^{-t}\, t^{\mu+m}\, L_\nu^\mu(zt)\, dt = -\frac{\sin \pi\nu}{\pi}\Gamma(\mu+\nu+1)\Gamma(\mu+m+1)(1-z)^\nu\\ &\qquad \times \sum_0^m{}_p \binom{m}{p}\frac{\Gamma(p-\nu)}{\Gamma(p+\mu+1)}\left(\frac{z}{1-z}\right)^p,\\ &\qquad R(z) < 1, \quad m \geq 0, \quad R(\mu+m) > -1,\\ &\text{or}\quad R(z) = 1, \quad R(\nu) > m \geq 0, \quad R(\mu+m) > -1.\end{aligned}\right\} \tag{2.18}$$

From (2.15),

$$\left.\begin{aligned}&\int_0^\infty e^{-t}\, t^\lambda\, L_\nu^\mu(t)\, dt = \frac{\Gamma(\lambda+1)\Gamma(\mu+\nu-\lambda)}{\Gamma(\nu+1)\Gamma(\mu-\lambda)},\\ &R(\mu+\nu-\lambda) > 0, \quad R(\lambda) > -1.\end{aligned}\right\} \tag{2.19}$$

From (2.15), (2.16), since the left of (2.15) converges uniformly in z to the right,

$$\int_0^\infty e^{-t}t^\lambda(zt)^{-\frac{1}{2}\mu}J_\mu(2\sqrt{zt})\,dt = \frac{\Gamma(\lambda+1)}{\Gamma(\mu+1)}\lim_{s\to\infty}F\left(-\frac{1}{s},\lambda+1;\mu+1;sz\right)$$

$$= \frac{\Gamma(\lambda+1)}{\Gamma(\mu+1)}F(\lambda+1;\mu+1;-z) = -\frac{\pi}{\sin\pi\lambda}\frac{1}{\Gamma(\mu-\lambda)}L^\mu_{-\lambda-1}(-z)$$

$$= +\frac{\pi}{\sin\pi(\mu-\lambda)}\frac{1}{\Gamma(\mu-\lambda)}e^{-z}L^\mu_{\lambda-\mu}(z) = \Gamma(\lambda-\mu+1)e^{-z}L^\mu_{\lambda-\mu}(z),$$

from (2.4). Therefore, setting $\lambda = \mu + \nu$,

$$L^\mu_\nu(z) = \frac{z^{-\frac{1}{2}\mu}e^z}{\Gamma(\nu+1)}\int_0^\infty e^{-t}t^{\frac{1}{2}\mu+\nu}J_\mu(2\sqrt{zt})\,dt, \qquad R(\mu+\nu) > -1. \tag{2.20}$$

A special case ($\nu = n \geq 0$) of this was given by Erdélyi in Math. Ann. 113 (1936) 357–362.

Since the series (2.1) is uniformly convergent for $|z|$ bounded, from (2.18),

$$\int_0^\infty e^{-t}t^\mu L^\mu_\nu(zt)L^\mu_{\nu'}(z't)\,dt$$

$$= -\frac{\sin\pi\nu'}{\pi}\Gamma(\mu+\nu'+1)\sum_0^\infty{}_n\frac{\Gamma(n-\nu')}{\Gamma(n+\mu+1)}\int_0^\infty e^{-t}t^{\mu+n}L^\mu_\nu(zt)\,dt\frac{z'^n}{n!}$$

$$= \frac{\sin\pi\nu}{\pi}\frac{\sin\pi\nu'}{\pi}\Gamma(\mu+\nu+1)\Gamma(\mu+\nu'+1)(1-z)^\nu$$

$$\times\sum_0^\infty{}_n\Gamma(n-\nu')z'^n\sum_0^n{}_p\frac{1}{p!(n-p)!}\frac{\Gamma(p-\nu)}{\Gamma(p+\mu+1)}\left(\frac{z}{1-z}\right)^p$$

$$= \frac{\sin\pi\nu}{\pi}\frac{\sin\pi\nu'}{\pi}\Gamma(\mu+\nu+1)\Gamma(\mu+\nu'+1)(1-z)^\nu$$

$$\times\sum_0^\infty{}_p\frac{\Gamma(p-\nu)}{p!\Gamma(p+\mu+1)}\left(\frac{zz'}{1-z}\right)^p\sum_0^\infty{}_q\Gamma(q+p-\nu')\frac{z'^q}{q!}$$

$$= \frac{\sin\pi\nu}{\pi}\frac{\sin\pi\nu'}{\pi}\Gamma(\mu+\nu+1)\Gamma(\mu+\nu'+1)(1-z)^\nu(1-z')^{\nu'}$$

$$\times\sum_0^\infty{}_p\frac{\Gamma(p-\nu)\Gamma(p-\nu')}{p!\Gamma(p+\mu+1)}\left[\frac{zz'}{(1-z)(1-z')}\right]^p$$

$$= \frac{\Gamma(\mu+\nu+1)\Gamma(\mu+\nu'+1)}{\Gamma(\mu+1)\Gamma(\nu+1)\Gamma(\nu'+1)}(1-z)^\nu(1-z')^{\nu'}$$

$$\times F\left(-\nu,-\nu';\mu+1;\frac{zz'}{(1-z)(1-z')}\right).$$

This result, clearly valid when $R(z) < \frac{1}{2}$, $R(z') < \frac{1}{2}$, $|z'| < 1$, may be extended analytically. Thus,

$$\left.\begin{aligned}\int_0^\infty e^{-t} t^\mu L_n^\mu(zt) L_{n'}^\mu(z't)\, dt &= \frac{\Gamma(\mu+n+1)\Gamma(\mu+n'+1)}{\Gamma(\mu+1)n!(n')!} \\ \times (1-z)^n (1-z')^{n'} F&\left(-n, -n'; \mu+1; \frac{zz'}{(1-z)(1-z')}\right), \\ &\text{where } R(\mu) > -1, \quad n \geq 0, \quad n' \geq 0,\end{aligned}\right\} \tag{2.21}$$

$$\left.\begin{aligned}\int_0^\infty e^{-t} t^\mu L_\nu^\mu(zt) L_n^\mu(z't)\, dt &= \frac{\Gamma(\mu+\nu+1)\Gamma(\mu+n+1)}{\Gamma(\mu+1)\Gamma(\nu+1)n!} \\ \times (1-z)^\nu (1-z')^n F&\left(-\nu, -n; \mu+1; \frac{zz'}{(1-z)(1-z')}\right), \\ &\text{where } R(z) < 1, \quad n \geq 0, \quad R(\mu) > -1, \\ &\text{or } R(z) = 1, \quad R(\nu) > n \geq 0, \quad R(\mu) > -1,\end{aligned}\right\} \tag{2.22}$$

$$\left.\begin{aligned}\int_0^\infty e^{-t} t^\mu L_\nu^\mu(zt) L_{\nu'}^\mu(z't)\, dt &= \frac{\Gamma(\mu+\nu+1)\Gamma(\mu+\nu'+1)}{\Gamma(\mu+1)\Gamma(\nu+1)\Gamma(\nu'+1)} \\ \times (1-z)^\nu (1-z')^{\nu'} F&\left(-\nu, -\nu'; \mu+1; \frac{zz'}{(1-z)(1-z')}\right), \\ &\text{where } R(z+z') < 1, \quad R(z) < 1, \quad R(z') < 1, \\ R(\mu) > -1, \quad &\text{or } R(z+.z') \leq 1, \quad R(z) \leq 1, \quad R(z') \leq 1, \\ &R(\mu+\nu+\nu') > -1, \quad R(\mu) > -1.\end{aligned}\right\} \tag{2.23}$$

From (2.22), setting $n = 0$,

$$\int_0^\infty e^{-t} t^\mu L_\nu^\mu(zt)\, dt = \frac{\Gamma(\mu+\nu+1)}{\Gamma(\nu+1)} (1-z)^\nu, \tag{2.24}$$

$$R(z) < 1, \quad R(\mu) > -1, \quad \text{or} \quad R(z) = 1, \quad R(\nu) > 0, \quad R(\mu) > -1.$$

From (2.21), and §14.51 of W. and W.,

$$\left.\begin{aligned}\int_0^\infty e^{-t} t^\mu L_n^\mu(zt) L_{n'}^\mu(t)\, dt &= \frac{\Gamma(\mu+n+1)}{(n-n')!(n')!} z^{n'} (1-z)^{n-n'}, \\ &n \geq 0, \quad n' \geq 0, \quad R(\mu) > -1.\end{aligned}\right\} \tag{2.25}$$

From this, for $n \geq 0$, $n' \geq 0$, $R(\mu) > -1$,

$$\left.\begin{aligned}\int_0^\infty e^{-t} t^\mu L_n^\mu(t) L_{n'}^\mu(t)\, dt &= 0, \qquad n' \neq n, \\ &= \frac{\Gamma(\mu+n+1)}{n!}, \quad n' = n.\end{aligned}\right\} \tag{2.26}$$

From (2.22) and §14.51 of W. and W, when ν is not an integer,

$$\left.\begin{aligned}\int_0^\infty e^{-t}t^\mu L_\nu^\mu(zt)L_n^\mu(t)\,dt &= -\frac{\sin\pi\nu}{\pi n!}\,\Gamma(\mu+\nu+1)\Gamma(n-\nu)\\ &\quad\times(-z)^n(1-z)^{\nu-n}, \qquad R(z)<1, \qquad n\geq 0, \qquad R(\mu)>-1,\\ &\qquad\text{or}\quad R(z)=1, \qquad R(\nu)>n\geq 0, \qquad R(\mu)>-1,\end{aligned}\right\}\tag{2.27}$$

and

$$\int_0^\infty e^{-t}t^\mu L_\nu^\mu(t)L_n^\mu(zt)\,dt = 0, \qquad R(\nu)>n\geq 0, \qquad R(\mu)>-1.$$

From (2.23) and §14.51 of W. and W., when ν, ν', $\nu-\nu'$ are not integers, and $|\arg(-z)|<\pi$,

$$\left.\begin{aligned}&\int_0^\infty e^{-t}t^\mu L_\nu^\mu(zt)L_{\nu'}^\mu(t)\,dt\\ &\qquad=-\frac{\sin\pi\nu}{\pi}\,\frac{\Gamma(\mu+\nu'+1)\Gamma(\nu'-\nu)}{\Gamma(\nu'+1)}\,(-z)^{\nu'}(1-z)^{\nu-\nu'},\\ &\qquad\qquad R(z)\leq 0, \qquad R(\mu+\nu+\nu')>-1,\\ &\qquad\qquad R(\nu)<R(\nu'), \qquad R(\mu)>-1.\end{aligned}\right\}\tag{2.28}$$

From (2.22), (2.23), and the uniform convergence of (2.15), and analytic continuation,

$$\left.\begin{aligned}&\int_0^\infty e^{-t}t^{\frac{1}{2}\mu}L_\nu^\mu(zt)J_\mu(2\sqrt{z't})\,dt = z'^{\frac{1}{2}\mu}(1-z)^\nu e^{-z'}L_\nu^\mu\left(\frac{-zz'}{1-z}\right),\\ &\text{where } R(\mu)>-1, \text{ and also } \nu=n\geq 0, z \text{ unrestricted, or } R(z)<1,\\ &\qquad\text{or}\quad R(z)=1, \qquad z\neq 1, \qquad z'>0, \qquad R(\mu+2\nu)>-\tfrac{1}{2},\\ &\qquad\qquad\text{or}\quad z=1, \qquad z'>0, \qquad R(\mu+2\nu)>\tfrac{1}{2}.\end{aligned}\right\}\tag{2.29}$$

From (2.29) and (2.13^1),

$$\left.\begin{aligned}\int_0^\infty e^{-t}t^{\frac{1}{2}\mu}J_\mu(2\sqrt{zt})L_\nu^\mu(t)\,dt &= \frac{1}{\Gamma(\nu+1)}\,z^{\frac{1}{2}\mu+\nu}e^{-z},\\ &\qquad\qquad \nu=n\geq 0, \qquad R(\mu)>-1,\\ &\qquad\text{or}\quad z=x>0, \qquad R(\mu+2\nu)>\tfrac{1}{2}, \qquad R(\mu)>-1.\end{aligned}\right\}\tag{2.30}$$

From (2.29) and (2.15),

$$\int_0^\infty e^{-t}J_\mu(2\sqrt{zt})J_\mu(2\sqrt{z't})\,dt = e^{-z-z'}I_\mu(2\sqrt{zz'}), \quad R(\mu)>-1.\tag{2.31}$$

From (2.23),

$$\left.\begin{aligned}\int_0^\infty e^{-t}\, t^\mu\, L_\nu^\mu(zt) L_{\nu'}^\mu(z't)\, dt &= \frac{\Gamma(\mu+\nu+\nu'+1)}{\Gamma(\nu+1)\Gamma(\nu'+1)}\, z^{\nu'}\, z''^{\nu}, \\ \text{where}\quad z+z'=1,\quad & R(z)\le 1,\quad R(z')\le 1, \\ & R(\mu+\nu+\nu')>-1,\quad R(\mu)>-1.\end{aligned}\right\} \tag{2.32}$$

Now
$$z^{-\alpha}\frac{\Gamma(z+\alpha)}{\Gamma(z)} = 1 + \frac{\alpha(\alpha-1)}{2z} + O\left(\frac{1}{z^2}\right). \tag{2.33}$$

Therefore

$$\left|z^{-\alpha}\frac{\Gamma(z+\alpha)}{\Gamma(z)} - \zeta^{-\alpha}\frac{\Gamma(\zeta+\alpha)}{\Gamma(\zeta)}\right| = \tfrac{1}{2}\,|\alpha(\alpha-1)|\left|\frac{1}{z}-\frac{1}{\zeta}\right| + O\left(\frac{1}{|z|^2}+\frac{1}{|\zeta|^2}\right).$$

Therefore

$$\lim_{z\to\infty} z^{-\alpha}\frac{\Gamma(z+\alpha)}{\Gamma(z)} = 1, \tag{2.34}$$

uniformly in α, for α bounded.

From (2.34), (2.5), (2.15),

$$\sum_0^\infty{}_n \frac{x^n}{\Gamma(n+\mu+1)}\, L_n^\mu(z) = e^z (xz)^{-\frac{1}{2}\mu}\, J_\mu(2\sqrt{xz}). \tag{2.35}$$

From (2.5), (2.13), and from Abel's theorem,

$$x^\nu = \frac{\Gamma(\mu+\nu+1)}{\Gamma(-\nu)}\sum_0^\infty{}_n \frac{\Gamma(n-\nu)}{\Gamma(n+\mu+1)}\, L_n^\mu(x), \tag{2.36}$$
$$x>0, \qquad R(\mu+2\nu)>-\tfrac{1}{2}.$$

Making use of the fact that

$$\frac{d^n}{dx^n}\, x^\alpha = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-n)}\, x^{\alpha-n}, \tag{2.37}$$

from (2.6),

$$\frac{h^{-\mu}}{n!}\frac{d^n}{dh^n}\left[h^{\mu+n}(1-h)^{-\mu-1}\, e^{-hz/(1-h)}\right] = (1-h)^{-\mu-n-1}\, e^{-hz/(1-h)}\, L_n^\mu\left(\frac{hz}{1-h}\right), \tag{2.38}$$
$$|h|<1, \qquad n\ge 0.$$

In the same manner, using (2.4),

$$L_n^\mu(z) = \lim_{\nu\to n}\frac{\sin\pi\nu}{\sin\pi(\mu+\nu)}\, e^z\, L_{-\mu-\nu-1}^\mu(-z) = \frac{1}{n!}\, e^z \sum_0^\infty{}_p \frac{\Gamma(p+\mu+n+1)}{\Gamma(p+\mu+1)}\,\frac{(-z)^p}{p!}$$

so
$$= \frac{1}{n!}\, z^{-\mu}\, e^z \frac{d^n}{dz^n}\left[z^{\mu+n}\sum_0^\infty{}_p \frac{(-z)^p}{p!}\right],$$

$$L_n^\mu(z) = \frac{1}{n!}\, z^{-\mu}\, e^z \frac{d^n}{dz^n}\left[z_0^{\mu+n}\, e^{-z}\right], \qquad n\ge 0. \tag{2.39}$$

From (2.1), for $m \geq 0$,

$$L_\nu^{-m}(z) = -\frac{\sin \pi\nu}{\pi}\,\Gamma(\nu - m + 1)\sum_m^\infty{}_p \frac{\Gamma(p-\nu)}{(p-m)!}\frac{z^p}{p!}$$

$$= -\frac{\sin \pi\nu}{\pi}\,\Gamma(\nu - m + 1) z^m \sum_0^\infty{}_p \frac{\Gamma(p+m-\nu)}{(p+m)!}\frac{z^p}{p!}$$

$$= \frac{\Gamma(\nu - m + 1)}{\Gamma(\nu + 1)}\,(-z)^m\, L_{\nu-m}^m(z).$$

That is,

$$L_\nu^{-m}(z) = -\frac{\sin \pi\nu}{\pi}\,\Gamma(\nu - m + 1)(-z)^m \sum_0^\infty{}_p \frac{\Gamma(p+m-\nu)}{(p+m)!}\frac{z^p}{p!}, \quad m \geq 0, \tag{2.40}$$

and

$$L_\nu^m(z) = \frac{\Gamma(\nu + m + 1)}{\Gamma(\nu + 1)}\,(-z)^{-m}\, L_{\nu+m}^{-m}(z). \tag{2.41}$$

That (2.41) holds for $m < 0$ as well, follows by replacing ν by $\nu - m$.

From (2.7), for $|h| < 1$,

$$\sum_0^\infty{}_n h^n L_n^{\mu+\mu'+1}(z + z') = (1-h)^{-\mu-\mu'-2}\, e^{-h(z+z')/(1-h)}$$

$$= \sum_0^\infty{}_p h^p L_p^\mu(z) \sum_0^\infty{}_q h^q L_q^{\mu'}(z') = \sum_0^\infty{}_n h^n \sum_0^n{}_p L_p^\mu(z) L_{n-p}^{\mu'}(z'),$$

so

$$L_n^{\mu+\mu'+1}(z + z') = \sum_0^n{}_p L_p^\mu(z) L_{n-p}^{\mu'}(z'), \qquad n \geq 0. \tag{2.42}$$

From (2.1),

$$\frac{d}{dz}\,[z^{+\mu} L_\nu^\mu(z)] = -\frac{\sin \pi\nu}{\pi}\,\Gamma(\mu + \nu + 1)\sum_0^\infty{}_p \frac{\Gamma(p-\nu)}{\Gamma(p+\mu)}\frac{z^{p+\mu-1}}{p!}$$

$$= (\mu + \nu) z^{\mu-1} L_\nu^{\mu-1}(z),$$

or

$$\frac{d}{dz}\,[z^\mu L_\nu^\mu(z)] = (\mu + \nu) z^{\mu-1} L_\nu^{\mu-1}(z). \tag{2.43}$$

From (2.1),

$$L_\nu^\mu(z) - L_\nu^{\mu+1}(z) + L_{\nu-1}^{\mu+1}(z) = -\frac{\sin \pi\nu}{\pi}\,\Gamma(\mu + \nu + 1)\sum_0^\infty{}_p \left\{\frac{\Gamma(p-\nu)}{\Gamma(p+\mu+1)}\right.$$

$$\left. - (\mu + \nu + 1)\frac{\Gamma(p-\nu)}{\Gamma(p+\mu+2)} - \frac{\Gamma(p-\nu+1)}{\Gamma(p+\mu+2)}\right\}\frac{z^p}{p!}$$

$$= -\frac{\sin \pi\nu}{\pi}\,\Gamma(\mu + \nu + 1)\sum_0^\infty{}_p \frac{\Gamma(p-\nu)}{\Gamma(p+\mu+2)}$$

$$\times \{p + \mu + 1 - \mu - \nu - 1 - p + \nu\}\frac{z^p}{p!} = 0,$$

or

$$L_\nu^\mu(z) - L_\nu^{\mu+1}(z) + L_{\nu-1}^{\mu+1}(z) = 0. \tag{2.44}$$

Also,

$$zL_{\nu-1}^{\mu+1}(z) + (z-\mu)L_\nu^\mu(z) + (\nu+1)L_{\nu+1}^{\mu-1}(z)$$

$$= \frac{\sin\pi\nu}{\pi}\Gamma(\mu+\nu+1)\sum_0^\infty{}_p \left\{ z\frac{\Gamma(p-\nu+1)}{\Gamma(p+\mu+2)} + (\mu-z)\frac{\Gamma(p-\nu)}{\Gamma(p+\mu+1)} + (\nu+1)\frac{\Gamma(p-\nu-1)}{\Gamma(p+\mu)}\right\}\frac{z^p}{p!}$$

$$= \frac{\sin\pi\nu}{\pi}\Gamma(\mu+\nu+1)\sum_0^\infty{}_p \frac{\Gamma(p-\nu-1)}{\Gamma(p+\mu+2)}\{z(p-\nu)(p-\nu-1) + (\mu-z)(p-\nu-1)(p+\mu+1) + (\nu+1)(p+\mu)(p+\mu+1)\}\frac{z^p}{p!}$$

$$= \frac{\sin\pi\nu}{\pi}\Gamma(\mu+\nu+1)\sum_0^\infty{}_p \frac{\Gamma(p-\nu-1)}{\Gamma(p+\mu+2)}\{-z(\mu+\nu+1)(p-\nu-1) + p(\mu+\nu+1)(p+\mu+1)\}\frac{z^p}{p!}$$

$$= \frac{\sin\pi\nu}{\pi}\Gamma(\mu+\nu+2)\left\{-\sum_0^\infty{}_p \frac{\Gamma(p-\nu)}{\Gamma(p+\mu+2)}\frac{z^{p+1}}{p!} + \sum_1^\infty{}_p \frac{\Gamma(p-\nu-1)}{\Gamma(p+\mu+1)}\frac{z^p}{(p-1)!}\right\} = 0,$$

or

$$zL_{\nu-1}^{\mu+1}(z) + (z-\mu)L_\nu^\mu(z) + (\nu+1)L_{\nu+1}^{\mu-1}(z) = 0. \tag{2.45}$$

From (2.43), (2.44), (2.45),

$$(\mu+\nu)L_{\nu-1}^\mu(z) + (z-\mu-2\nu-1)L_\nu^\mu(z) + (\nu+1)L_{\nu+1}^\mu(z) = 0, \tag{2.46}$$

$$(\mu+\nu)L_\nu^{\mu-1}(z) - (z+\mu)L_\nu^\mu(z) + zL_\nu^{\mu+1}(z) = 0, \tag{2.47}$$

$$zL_\nu^\mu(z) = -(\nu+1)L_{\nu+1}^{\mu-1}(z) + (\mu+\nu)L_\nu^{\mu-1}(z), \tag{2.48}$$

$$zL_\nu^\mu(z) = (z-\nu)L_\nu^{\mu-1}(z) + (\mu+\nu-1)L_{\nu-1}^{\mu-1}(z), \tag{2.49}$$

$$(\mu+\nu+1)L_\nu^\mu(z) = (\nu+1)L_{\nu+1}^{\mu+1}(z) + (z-\nu-1)L_\nu^{\mu+1}(z), \tag{2.50}$$

$$L_\nu^\mu(z) = L_\nu^{\mu+1}(z) - L_{\nu-1}^{\mu+1}(z), \tag{2.51}$$

$$\nu L_\nu^\mu(z) = -zL_{\nu-1}^{\mu+1}(z) + (\mu+\nu)L_{\nu-1}^\mu(z), \tag{2.52}$$

$$\nu L_\nu^\mu(z) = (\nu-z)L_{\nu-1}^\mu(z) + (\mu+\nu-1)L_{\nu-1}^{\mu-1}(z), \tag{2.53}$$

$$(\mu+\nu+1)L_\nu^\mu(z) = zL_{\nu+1}^{\mu+1}(z) + (\nu+1-z)L_{\nu+1}^\mu(z), \tag{2.54}$$

$$L_\nu^\mu(z) = L_{\nu-1}^\mu(z) - L_{\nu-1}^{\mu-1}(z), \tag{2.55}$$

$$z\frac{d}{dz}L_\nu^\mu(z) = (\nu+1)L_{\nu+1}^\mu(z) + (z-\mu-\nu-1)L_\nu^\mu(z), \tag{2.56}$$

$$z\frac{d}{dz}L_\nu^\mu(z) = \nu L_\nu^\mu(z) - (\mu+\nu)L_{\nu-1}^\mu(z), \tag{2.57}$$

$$\frac{d}{dz}L_\nu^\mu(z) = -L_\nu^{\mu+1}(z) + L_\nu^\mu(z), \tag{2.58}$$

$$z\frac{d}{dz}L_\nu^\mu(z) = -\mu L_\nu^\mu(z) + (\mu+\nu)L_\nu^{\mu-1}(z), \tag{2.59}$$

$$\frac{d}{dz}L_\nu^\mu(z) = -L_{\nu-1}^{\mu+1}(z). \tag{2.60}$$

From (2.45),

$$zL_{\nu-3}^{\mu+2}(z) - (\mu + 1 - z)L_{\nu-1}^{\mu+1}(z) + \nu L_\nu^\mu(z) = 0.$$

Therefore, from (2.60),

$$z\frac{d^2}{dz^2}L_\nu^\mu(z) + (\mu + 1 - z)\frac{d}{dz}L_\nu^\mu(z) + \nu L_\nu^\mu(z) = 0, \tag{2.61}$$

so $L_\nu^\mu(z)$ is a solution of (1.12).

We now desire an expression for $L_\nu^\mu(z)$ when ν is large. This might be obtained, as Erdélyi has done, by applying the method of steepest descent to the integral (2.20). We shall, however, use a different method.

Let y satisfy

$$zy'' + (\mu + 1 - z)y' + \nu y = 0. \tag{2.62}$$

Write

$$y = e^{2\alpha z^{\frac{1}{2}}}u.$$

$$y' = e^{2\alpha z^{\frac{1}{2}}}(u' + \alpha z^{-\frac{1}{2}}u), \qquad y'' = e^{2\alpha z^{\frac{1}{2}}}[u'' + 2\alpha z^{-\frac{1}{2}}u' + (\alpha^2 z^{-1} - \tfrac{1}{2}\alpha z^{-\frac{3}{2}})u].$$

Therefore

$$zu'' + (\mu + 1 + 2\alpha z^{\frac{1}{2}} - z)u' + [(\mu + \tfrac{1}{2})\alpha z^{-\frac{1}{2}} + \alpha^2 + \nu - \alpha z^{\frac{1}{2}}]u = 0.$$

Let

$$\alpha = \pm i\nu^{\frac{1}{2}}.$$

Then

$$zu'' + (\mu + 1 \pm 2i\nu^{\frac{1}{2}}z^{\frac{1}{2}} - z)u' \pm i\nu^{\frac{1}{2}}[(\mu + \tfrac{1}{2})z^{-\frac{1}{2}} - z^{\frac{1}{2}}]u = 0.$$

Suppose

$$u = \nu^\sigma[\mu_0 + \mu_1\nu^{-\frac{1}{2}} + \mu_2\nu^{-1} + \mu_3\nu^{-\frac{3}{2}} + \cdots],$$

where $\mu_0, \mu_1, \mu_2, \cdots$ are independent of ν. Then

$$z\mu_0'' + z\mu_1''\nu^{-\frac{1}{2}} + \cdots + (\mu + 1 \pm 2i\nu^{\frac{1}{2}}z^{\frac{1}{2}} - z)(\mu_0' + \mu_1'\nu^{-\frac{1}{2}} + \cdots)$$
$$\pm i\nu^{\frac{1}{2}}[(\mu + \tfrac{1}{2})z^{-\frac{1}{2}} - z^{\frac{1}{2}}](\mu_0 + \mu_1\nu^{-\frac{1}{2}} + \cdots) = 0.$$

Equating the coefficient of $\nu^{\frac{1}{2}}$ to zero,

$$2\mu_0' + \left(\frac{\mu + \frac{1}{2}}{z} - 1\right)\mu_0 = 0,$$

so

$$\mu_0 = (\text{const.})z^{-\frac{1}{2}\mu - \frac{1}{4}}e^{\frac{1}{2}z}.$$

Therefore, for A, B independent of ν,

$$y = \nu^\sigma z^{-\frac{1}{2}\mu - \frac{1}{4}}e^{\frac{1}{2}z}(Ae^{2i\sqrt{\nu z}} + Be^{-2i\sqrt{\nu z}})[1 + O(\nu^{-\frac{1}{2}})]. \tag{2.63}$$

In particular

$$L^{\mu}_{\nu}(z) = \nu^{\nu} z^{-\frac{1}{2}\mu-\frac{1}{4}} e^{\frac{1}{2}z}(Ae^{2i\sqrt{\nu z}} + Be^{-2i\sqrt{\nu z}})[1 + 0(\nu^{-1})].$$

From (2.15) it follows that $\sigma = \frac{1}{2}\mu - \frac{1}{4}$, and that

$$(\nu z)^{\frac{1}{2}} J_{\mu}(2\sqrt{\nu z}) = (Ae^{2i\sqrt{\nu z}} + Be^{-2i\sqrt{\nu z}})[1 + 0(\nu^{-1})]$$

From §17.5, W. and W.,

$$Ae^{2i\sqrt{\nu z}} + Be^{-2i\sqrt{\nu z}} = \frac{1}{\sqrt{\pi}} \cos(2\sqrt{\nu z} - \tfrac{1}{2}\pi\mu - \tfrac{1}{4}\pi),$$

so

$$L^{\mu}_{\nu}(z) = \frac{1}{\sqrt{\pi}} \nu^{\frac{1}{2}\mu-\frac{1}{4}} z^{-\frac{1}{2}\mu-\frac{1}{4}} e^{\frac{1}{2}z} \cos(2\sqrt{\nu z} - \tfrac{1}{2}\pi\mu - \tfrac{1}{4}\pi)[1 + 0(\nu^{-1})] \quad (2.64)$$

From (2.61), when $R(\mu) > -1$.

$$(\nu - \nu') \int_0^z e^{-t} t^{\mu} L^{\mu}_{\nu}(t) L^{\mu}_{\nu'}(t)\, dt = z^{\mu+1} e^{-z}[L^{\mu}_{\nu}(z) L^{\mu\prime}_{\nu'}(z) - L^{\mu}_{\nu'}(z) L^{\mu\prime}_{\nu}(z)] \quad (2.65)$$

From this and (2.4), when $R(\mu) > -1$,

$$\left.\begin{aligned}(\mu + \nu + \nu' + 1) \int_0^z t^{\mu} L^{\mu}_{\nu}(t) L^{\mu}_{\nu'}(-t)\, dt = z^{\mu}\{&(z - \nu + \nu') L^{\mu}_{\nu}(z) L^{\mu}_{\nu'}(-z) \\ &+ (\mu + \nu) L^{\mu}_{\nu-1}(z) L^{\mu}_{\nu'}(-z) - (\mu + \nu') L^{\mu}_{\nu}(z) L^{\mu}_{\nu'-1}(-z)\}\end{aligned}\right\} \quad (2.66)$$

3. *The Function* $U^{\mu}_{\nu}(z)$.

Let us write

$$T^{\mu}_{\nu}(z) = z^{-\mu} L^{-\mu}_{\mu+\nu}(z).$$

From (2.43),

$$\frac{d}{dz} T^{\mu}_{\nu}(z) = \nu T^{\mu+1}_{\nu-1}(z).$$

From (2.47),

$$(\nu - 1) L^{-\mu-2}_{\mu+2+\nu-3}(z) + (\mu + 1 - z) L^{-\mu-1}_{\mu+1+\nu-1}(z) + z L^{-\mu}_{\mu+\nu}(z) = 0,$$

so

$$z \frac{d^2}{dz^2} T^{\mu}_{\nu}(z) + (\mu + 1 - z) \frac{d}{dz} T^{\mu}_{\nu}(z) + \nu T^{\mu}_{\nu}(z) = 0,$$

so $T^{\mu}_{\nu}(z)$ satisfies (1.12). Therefore

$$U^{\mu}_{\nu}(z) = \pm i \csc \pi\mu \left\{ e^{\mp \pi i \mu} L^{\mu}_{\nu}(z) - \frac{\Gamma(\mu + \nu + 1)}{\Gamma(\nu + 1)} z^{-\mu} L^{-\mu}_{\mu+\nu}(z) \right\}, \quad (3.1)$$

according as $0 < \arg z < \pi$, $\quad -\pi < \arg z < 0$,

satisfies

$$z \frac{d^2}{dz^2} U^{\mu}_{\nu}(z) + (\mu + 1 - z) \frac{d}{dz} U^{\mu}_{\nu}(z) + \nu U^{\mu}_{\nu}(z) = 0. \quad (3.2)$$

Also $U_\nu^\mu(z)$ satisfies the recurrence formulae (2.43)–(2.60). It is analytic everywhere except on the real axis in the z-plane, and forms a second solution of (1.12), linearly independent of $L_\nu^\mu(z)$. We might have added the positive real axis to our region of analyticity by taking a single rather than a double sign in (3.1), but this definition is more symmetrical and gives rise to a simpler and better asymptotic expansion. We shall use this same double sign convention hereafter, unless otherwise noted.

From (2.41), (3.1) becomes indeterminate when μ is an integer, so in this case we must take the limit of the expression for $U_\nu^\mu(z)$ as μ approaches an integer m, say. We may write

$$U_\nu^\mu(z) = \mp i \frac{\sin \pi\nu}{\pi \sin \pi\mu} \Gamma(\mu + \nu + 1) \left\{ e^{\mp \pi i \mu} \sum_0^\infty{}_p \frac{\Gamma(p - \nu)}{\Gamma(p + \mu + 1)} \frac{z^p}{p!} - \frac{\sin \pi(\mu + \nu)}{\sin \pi\nu} z^{-\mu} \sum_0^\infty{}_p \frac{\Gamma(p - \mu - \nu)}{\Gamma(p - \mu + 1)} \frac{z^p}{p!} \right\} \tag{3.3}$$

Now let $\mu = m + \epsilon$, $m \geq 0$, $|\epsilon| \ll 1$. Then

$$\sin \pi\mu \simeq (-)^m \pi\epsilon, \qquad \frac{1}{\Gamma(p + \mu + 1)} \simeq \frac{1}{(p + m)!} \left[1 + \epsilon \left(\gamma - \sum_1^{p+m}{}_k \frac{1}{k} \right) \right],$$

$$z^{-\mu} \simeq z^{-m}(1 - \epsilon \ln z) \quad e^{\mp \pi i \mu} \simeq (-)^m (1 \mp \pi i \epsilon),$$

$$\frac{\sin \pi(\mu + \nu)}{\sin \pi\nu} \simeq (-)^m (1 + \pi\epsilon \cot \pi\nu),$$

$$\frac{1}{\Gamma(p - \mu + 1)} \simeq \frac{1}{(p - m)!} \left[1 - \epsilon \left(\gamma - \sum_1^{p-m}{}_k \frac{1}{k} \right) \right], \qquad p \geq m,$$

$$\simeq (-)^{m-p} \epsilon (m - p - 1)!, \qquad p < m,$$

$$\Gamma(p - \mu - \nu) \simeq \Gamma(p - m - \nu)[1 - \epsilon\psi(p - m - \nu)],$$

where

$$\psi(z) = \frac{d}{dz} \ln \Gamma(z). \tag{3.4}$$

Therefore

$$U_\nu^{m+\epsilon}(z) \simeq \mp i \frac{\sin \pi\nu}{\pi^2 \epsilon} \Gamma(m + \nu + 1) \left\{ \sum_0^\infty{}_p \frac{\Gamma(p - \nu)}{(p + m)!} \times \left[1 + \epsilon \left(\gamma \mp \pi i - \sum_1^{p+m}{}_k \frac{1}{k} \right) \right] \frac{z^p}{p!} - \epsilon \sum_0^{m-1}{}_p \Gamma(p - m - \nu)(m - p - 1)! \frac{(-z)^{p-m}}{p!} + \sum_m^\infty{}_p \frac{\Gamma(p - m - \nu)}{(p - m)!} \left[-1 + \epsilon \left\{ \ln z + \gamma - \sum_1^{p-m}{}_k \frac{1}{k} + \psi(p - m - \nu) - \pi \cot \pi\nu \right\} \right] \frac{z^{p-m}}{p!} \right.$$

Therefore

$$U_\nu^m(z) = \pm i \frac{\sin \pi\nu}{\pi^2} \Gamma(m+\nu+1) \sum_1^m{}_p \Gamma(-p-\nu) \frac{(p-1)!}{(m-p)!} (-z)^{p-m}$$

$$\mp i \frac{\sin \pi\nu}{\pi^2} \Gamma(m+\nu+1) \sum_0^\infty{}_p \frac{\Gamma(p-\nu)}{(p+m)!} \left[\ln z + 2\gamma \mp \pi i \right.$$

$$\left. - \sum_0^p{}_k \frac{1}{k} - \sum_1^{p+m}{}_k \frac{1}{k} + \psi(p-\nu) - \pi \cot \pi\nu \right] \frac{z^p}{p!}$$

Therefore, for $m \geq 0$,

$$\left.\begin{aligned} U_\nu^m(z) = {} & \mp \frac{i}{\pi} \sum_1^m{}_p \frac{\Gamma(m+\nu+1)}{\Gamma(p+\nu+1)} \frac{(p-1)!}{(m-p)!} z^{-p} \\ & \mp i \frac{\sin \pi\nu}{\pi^2} \Gamma(m+\nu+1) \sum_0^\infty{}_p \frac{\Gamma(p-\nu)}{(p+m)!} \left[\ln z + 2\gamma \mp \pi i \right. \\ & \left. - \sum_1^p \frac{1}{k} - \sum_1^{p+m}{}_k \frac{1}{k} + \psi(p-\nu) - \pi \cot \pi\nu \right] \frac{z^p}{p!}. \end{aligned}\right\} \quad (3.5)$$

From (3.1),

$$U_\nu^\mu(z) = \frac{\Gamma(\mu+\nu+1)}{\Gamma(\nu+1)} e^{\mp \pi i \mu} z^{-\mu} U_{\mu+\nu}^{-\mu}(z), \quad (3.6)$$

From this, for $m \geq 0$,

$$\left.\begin{aligned} U_\mu^{-m}(z) = {} & \pm \frac{(-)^{m-1}}{\pi} i \sum_0^{m-1}{}_p \frac{\Gamma(-m+\nu+1)}{\Gamma(-p+\nu+1)} \frac{(m-p-1)!}{p!} z^p \\ & \mp i \frac{\sin \pi\nu}{\pi_2} \Gamma(-m+\nu+1) \sum_m^\infty{}_p \frac{\Gamma(p-\nu)}{(p-m)!} \left[\ln z + 2\gamma \mp \pi i - \sum_1^p{}_k \frac{1}{k} \right. \\ & \left. - \sum_1^{p-m}{}_k \frac{1}{k} + \psi(p-\nu) - \pi \cot \pi\nu \right] \frac{z^p}{p!} \end{aligned}\right\} \quad (3.7)$$

From (3.5), (3.7),

$$\left.\begin{aligned} & U_\nu^m(z) \simeq \mp \frac{i}{\pi} \frac{(m-1)!}{z^m}, && m > 0, && |z| \ll 1, \\ & U_\nu^0(z) \simeq 1 \pm \frac{i}{\pi} [\ln z + 2\gamma + \psi(-\nu) - \pi \cot \pi\nu], && && |z| \ll 1, \\ & U_\nu^{-m}(z) \simeq \pm \frac{i}{\pi} (-)^{m-1} (m-1)! \frac{\Gamma(\nu-m+1)}{\Gamma(\nu+1)}, && m > 0, && \\ & && && |z| \ll 1. \end{aligned}\right\} \quad (3.8)$$

In connection with (3.5), (3.7), (3.8), we note that

$$\lim_{\nu \to n} [\psi(p - \nu) - \pi \cot \pi\nu] = -\gamma + \sum_1^{n-p} {}_k \frac{1}{k}, \qquad n \geq p \geq 0. \tag{3.9}$$

Therefore, for $m, n \geq 0$,

$$\left.\begin{aligned} U_n^m(z) = {} & \mp \frac{i}{\pi} \sum_1^m {}_p \frac{(m+n)!}{(p+n)!} \frac{(p-1)!}{(m-p)!} z^{-p} \pm \frac{i}{\pi} \frac{(m+n)!}{n!} \sum_0^n {}_p \binom{n}{p} \\ & \times \left[lnz + \gamma \mp \pi i + \sum_1^{n-p} {}_k \frac{1}{k} - \sum_1^p {}_k \frac{1}{k} - \sum_1^{p+m} {}_k \frac{1}{k} \right] \frac{(-z)^p}{(p+m)!} \\ & \pm \frac{i}{\pi} (-)^n (n+m)! \sum_{n+1}^\infty {}_p \frac{(p-n-1)!}{(p+m)!} \frac{z^p}{p!}, \qquad m, n \geq 0, \end{aligned}\right\} \tag{3.10}$$

$$\left.\begin{aligned} U_n^{-m}(z) = {} & \pm \frac{i}{\pi} (-)^{m-1} \sum_0^{m-1} {}_p \frac{(n-m)!}{(n-p)!} \frac{(m-p-1)!}{p!} z^p \\ & \pm \frac{i}{\pi} \frac{(n-m)!}{n!} \sum_m^n {}_p \binom{n}{p} \left[\ln z + \gamma \mp \pi i + \sum_1^{n-p} {}_k \frac{1}{k} - \sum_1^p {}_k \frac{1}{k} \right. \\ & \left. - \sum_1^{p-m} {}_k \frac{1}{k} \right] \frac{(-z)^p}{(p-m)!} \\ & \pm \frac{i}{\pi} (-)^n (n-m)! \sum_{n+1}^\infty {}_p \frac{(p-n-1)!}{(p-m)!} \frac{z^p}{p!}, \qquad n \geq m > 0, \end{aligned}\right\} \tag{3.11}$$

$$\lim_{\nu \to n} (\nu - n) U_\nu^{-m}(z) = \pm \frac{i}{\pi} \left\{ \frac{(-)^n}{n!} \sum_0^n {}_p \binom{n}{p} \frac{(m-p-1)!}{(m-n-1)!} z^p \right. \tag{3.12}$$

$$\left. + \frac{(-)^{m-1}}{(m-n-1)!} \sum_m^\infty {}_p \frac{(p-n-1)!}{(p-m)!} \frac{z^p}{p!} \right\}, \qquad 0 \leq n < m,$$

$$U_{-n}^m(z) = \pm \frac{i}{\pi} \left\{ - \sum_n^m {}_p \frac{(m-n)!}{(p-n)!} \frac{(p-1)!}{(m-p)!} z^{-p} \right. \tag{3.13}$$

$$\left. + (-)^n (m-n)! \sum_0^\infty {}_p \frac{(p+n-1)!}{(p+m)!} \frac{z^p}{p!}, \qquad 0 < n \leq m, \right.$$

$$\lim_{\nu \to -n} (\nu + n) U_\nu^m(z) = \pm \frac{i}{\pi} \frac{(-)^{m-1}}{(-m+n-1)!} \sum_0^\infty {}_p \frac{(p+n-1)!}{(p+m)!} \frac{z^p}{p!}, \tag{3.14}$$

$$0 \leq m < n,$$

$$\lim_{\nu \to -n} (\nu + n) U_\nu^{-m}(z) = \pm \frac{i}{\pi} \frac{(-)^{m-1}}{(m+n-1)!} \sum_m^\infty {}_p \frac{(p+n-1)!}{(p-m)!} \frac{z^p}{p!}, \tag{3.15}$$

$$m \geq 0, \qquad n > 0.$$

From these,

$$\left.\begin{aligned} &U_n^m(z) \simeq \mp \frac{i}{\pi}\frac{(m-1)!}{z^m}, \qquad m>0, \quad n \geq -m, \qquad |z| \ll 1,\\ &\lim_{\nu\to n}(\nu-n)U_\nu^m(z) \simeq \pm\frac{i}{\pi}(-)^{m-1}\frac{(-n-1)!}{m!(-n-m-1)!},\\ &\qquad\qquad n<-m<0, \qquad |z| \ll 1, \end{aligned}\right\} \tag{3.16}$$

$$\left.\begin{aligned} &U_n^0(z) \simeq 1 \pm \frac{i}{\pi}\left(lnz+\gamma+\sum_1^n{}_k\frac{1}{k}\right), \qquad n\geq 0, \qquad |z|\ll 1,\\ &\lim_{\nu\to n}(\nu-n)U_\nu^0(z) \simeq \mp\frac{i}{\pi}, \qquad n<0, \qquad |z|\ll 1, \end{aligned}\right\} \tag{3.19}$$

$$\left.\begin{aligned} &U_n^m(z) \simeq \pm\frac{i}{\pi}(-)^{m-1}(-m-1)!(n+m)!/n!, \qquad 0<-m\leq n,\\ &\qquad\qquad |z|\ll 1,\\ &\lim_{\nu\to n}(\nu-n)U_\nu^m(z) \simeq \pm\frac{i}{\pi}\frac{(-)^n}{n!}\frac{(-m-1)!}{(-m-n-1)}, \qquad 0\leq n<-m,\\ &\qquad\qquad |z|\ll 1,\\ &\lim_{\nu\to n}(\nu-n)U_\nu^m(z) \simeq \pm\frac{i}{\pi}(-)^{m-1}\frac{z^{-m}}{(-m)!} \qquad m\geq 0, \qquad n>0,\\ &\qquad\qquad |z|\ll 1. \end{aligned}\right\} \tag{3.18}$$

From (2.10), (3.1),

$$\begin{aligned} U_\nu^\mu(z) \sim{}& \frac{e^{\pm\pi i(\nu+\frac{1}{2})}}{\pi\Gamma(\nu+1)}z^{-\mu-\nu-1}\\ &\times e^z\sum_0^\infty{}_n\,\Gamma(n+\nu+1)\Gamma(n+\mu+\nu+1)\frac{1}{n!z^n}, \qquad |z|\gg 1. \end{aligned} \tag{3.19}$$

Therefore,

$$U_\nu^\mu(z) \sim \frac{1}{\pi}e^{\pm\pi i(\nu+\frac{1}{2})}\,\Gamma(\mu+\nu+1)z^{-\mu-\nu-1}e^z, \qquad |z|\gg 1. \tag{3.20}$$

From (2.16), (3.1),

$$\left.\begin{aligned} &\int_0^\infty e^{-t}\,t^\lambda\,U_\nu^\mu(zt)\,dt\\ &\quad = \pm i\csc\pi\mu\left\{e^{\mp\pi i\mu}\frac{\Gamma(\lambda+1)\Gamma(\mu+\nu+1)}{\Gamma(\mu+1)\Gamma(\nu+1)}F(-\nu,\lambda+1;\mu+1;z)\right.\\ &\qquad \left.-\frac{\Gamma(\lambda-\mu+1)}{\Gamma(-\mu+1)}z^{-\mu}F(-\mu-\nu,\lambda-\mu+1;-\mu+1;z)\right\},\\ &\text{when } R(z)<1,\ R(\lambda-\mu)>-1,\ R(\mu)\geq 0;\text{ or } R(z)<1,\\ &\qquad R(\lambda)>-1,\ R(\mu)\leq 0;\text{ or } R(z)=1,\ R(\lambda-\mu-\nu)>0,\\ &\qquad\qquad R(\lambda-\mu)>-1,\ R(\mu)\geq 0;\text{ or } R(z)=1,\ R(\lambda-\mu-\nu)>0,\\ &\qquad\qquad\qquad R(\lambda)>-1,\ R(\mu)\leq 0. \end{aligned}\right\} \tag{3.21}$$

From (3.1) and (2.15)

$$\left.\begin{aligned}\lim_{s\to 0}\left(\frac{s}{\nu}\right)^{\mu} U^{\mu}_{\nu/s}(sz) &= (\nu z)^{-\frac{1}{2}\mu} H^{(1)}_{\mu}(2\sqrt{\nu z}), \qquad I(z) > 0,\\ &= (\nu z)^{-\frac{1}{2}\mu} H^{(2)}_{\mu}(2\sqrt{\nu z}). \qquad I(z) < 0.\end{aligned}\right\} \tag{3.22}$$

From (2.20)

$$\left.\begin{aligned}U^{\mu}_{\nu}(z) &= \frac{z^{-\frac{1}{2}\mu} e^{z}}{\Gamma(\nu+1)} \int_0^{\infty} e^{-t} t^{\frac{1}{2}\mu+\nu} H^{(1)}_{\mu}(2\sqrt{zt})\, dt, \qquad I(z) > 0,\\ &= \frac{z^{-\frac{1}{2}\mu} e^{z}}{\Gamma(\nu+1)} \int_0^{\infty} e^{-t} t^{\frac{1}{2}\mu+\nu} H^{(2)}_{\mu}(2\sqrt{zt})\, dt, \qquad I(z) < 0.\end{aligned}\right\} \tag{3.23}$$

provided $R(\nu) > -1$ when $R(\mu) \geq 0$, or $R(\mu+\nu) > -1$ when $R(\mu) \leq 0$.

As in (2.64), from (2.63) and (3.22),

$$U^{\mu}_{\nu}(z) = \frac{1}{\sqrt{\pi}}\, \nu^{\frac{1}{2}\mu-\frac{1}{4}} z^{-\frac{1}{2}\mu-\frac{1}{4}} e^{\frac{1}{2}z} e^{\pm i(2\sqrt{\nu z}-\frac{1}{2}\nu\mu-\frac{1}{4}\nu)}[1 + O(\nu^{-\frac{1}{2}})]. \tag{3.24}$$

4. *Relations Between $L^{\mu}_{\nu}(z)$ and $U^{\mu}_{\nu}(z)$.*

Write

$$u = L^{\mu}_{\nu}(z), \qquad v = U^{\mu}_{\nu}(z).$$

Then

$$zu'' + (\mu + 1 - z)u' + \nu u = 0,$$
$$zv'' + (\mu + 1 - z)v' + \nu v = 0,$$

so

$$z(vu'' - uv'') + (\mu + 1 - z)(vu' - uv') = 0,$$
$$\frac{d(vu' - uv')}{vu' - uv'} + \left(\frac{\mu+1}{z} - 1\right) dz = 0$$
$$\therefore (vu' - uv')z^{\mu+1}e^{-z} = \text{const.}$$

Therefore

$$U^{\mu}_{\nu}(z)\frac{d}{dz}L^{\mu}_{\nu}(z) - L^{\mu}_{\nu}(z)\frac{d}{dz}U^{\mu}_{\nu}(z) = A^{\mu}_{\nu} z^{-\mu-1} e^{z},$$

or

$$U^{\mu}_{\nu}(z)L^{\mu}_{\nu+1}(z) - L^{\mu}_{\nu}(z)U^{\mu}_{\nu+1}(z) = \frac{A^{\mu}_{\nu}}{\nu+1}\, z^{-\mu} e^{z}.$$

Therefore, using (2.1),

$$A^{\mu}_{\nu} = \frac{\Gamma(\mu+\nu+1)}{\Gamma(\mu+1)\Gamma(\nu+1)} \lim_{z\to 0} z^{\mu}\{(\mu+\nu+1)U^{\mu}_{\nu}(z) - (\nu+1)U^{\mu}_{\nu+1}(z)\}$$
$$= \frac{\Gamma(\mu+\nu+1)}{\Gamma(\mu+1)\Gamma(\nu+1)} \lim_{z\to 0} z^{\mu+1} U^{\mu+1}_{\nu}(z).$$

From (3.3),

$$\lim_{z\to 0} z^{\mu} U_{\nu}^{\mu}(z) = \mp \frac{i}{\pi}\Gamma(\mu), \qquad R(\mu) > 0. \tag{4.1}$$

Therefore

$$A_{\nu}^{\mu} = \mp \frac{i}{\pi}\frac{\Gamma(\mu+\nu+1)}{\Gamma(\nu+1)}, \qquad R(\mu) > -1,$$

so

$$\begin{aligned} U_{\nu}^{\mu}(z)\frac{d}{dz}L_{\nu}^{\mu}(z) - L_{\nu}^{\mu}(z)\frac{d}{dz}U_{\nu}^{\mu}(z) \\ = \mp \frac{i}{\pi}\frac{\Gamma(\mu+\nu+1)}{\Gamma(\nu+1)} z^{-\mu-1} e^{z}, \qquad R(\mu) > -1, \end{aligned} \tag{4.2}$$

$$U_{\nu}^{\mu}(z)L_{\nu+1}^{\mu}(z) - L_{\nu}^{\mu}(z)U_{\nu+1}^{\mu} = \mp \frac{i}{\pi}\frac{\Gamma(\mu+\nu+1)}{\Gamma(\nu+1)} z^{-\mu} e^{z}, \qquad R(\mu) > -1. \tag{4.3}$$

From (2.46),

$$(\mu+\nu)L_{\nu-1}^{\mu}(z)U_{\nu}^{\mu}(t) + (z-\mu-2\nu-1)L_{\nu}^{\mu}(z)U_{\nu}^{\mu}(t) + (\nu+1)L_{\nu+1}^{\mu}(z)U_{\nu}^{\mu}(t) = 0,$$

$$(\mu+\nu)L_{\nu}^{\mu}(z)U_{\nu-1}^{\mu}(t) + (t-\mu-2\nu-1)L_{\nu}^{\mu}(z)U_{\nu}^{\mu}(t) + (\nu+1)L_{\nu}^{\mu}(z)U_{\nu+1}^{\mu}(t) = 0$$

Therefore

$$B_{\nu}^{\mu}(z,t) - B_{\nu-1}^{\mu}(z,t) = \frac{\Gamma(\nu+1)}{\Gamma(\mu+\nu+1)}(t-z)L_{\nu}^{\mu}(z)U_{\nu}^{\mu}(t), \tag{4.4}$$

where

$$B_{\nu}^{\mu}(z,t) = \frac{\Gamma(\nu+2)}{\Gamma(\mu+\nu+1)}\{L_{\nu+1}^{\mu}(z)U_{\nu}^{\mu}(t) - L_{\nu}^{\mu}(z)U_{\nu+1}^{\mu}(t)\}. \tag{4.5}$$

Therefore

$$B_{\nu+n}^{\mu}(z,t) - B_{\nu-1}^{\mu}(z,t) = (t-z)\sum_{0}^{n}{}_{p}\frac{\Gamma(p+\nu+1)}{\Gamma(p+\mu+\nu+1)}L_{\nu+p}^{\mu}(z)U_{\nu+p}^{\mu}(t). \tag{4.6}$$

From (2.64) and (3.24) we see that as $\nu \to \infty$, $B_{\nu}^{\mu}(z,t) \to 0$ and therefore

$$\left.\begin{aligned} \frac{B_{\nu-1}^{\mu}(z,t)}{t-z} = -\sum_{0}^{\infty}{}_{p}\frac{\Gamma(p+\nu+1)}{\Gamma(p+\mu+\nu+1)}L_{\nu+p}^{\mu}(z)U_{\nu+p}^{\mu}(t) \\ \text{when} \\ |z|^{\frac{1}{2}}\sin|\tfrac{1}{2}\arg z| < |t|^{\frac{1}{2}}\sin|\tfrac{1}{2}\arg t|, \qquad I(z), I(t) \neq 0. \end{aligned}\right\} \tag{4.7}$$

Now

$$B_{-1}^{\mu}(z,t) = \frac{1}{\Gamma(\mu)}\lim_{\nu\to 0}\{L_{\nu}^{\mu}(z)U_{\nu-1}^{\mu}(t) - L_{\nu-1}^{\mu}(z)U_{\nu}^{\mu}(t)\} = \frac{1}{\Gamma(\mu)}U_{-1}^{\mu}(t).$$

From (3.3)

$$U_{-1}^{\mu}(z) = \mp \frac{i}{\pi}\Gamma(\mu)z^{-\mu}e^{z}, \tag{4.8}$$

so

$$B^{\mu}_{-1}(z, t) = \mp\frac{i}{\pi} t^{-\mu} e^{t}, \tag{4.9}$$

so, from (4.7),

$$\left.\begin{aligned} \pm\frac{i}{\pi}\frac{t^{-\mu}e^{t}}{t-z} &= \sum_{0}^{\infty}{}_{p} \frac{p!}{\Gamma(p+\mu+1)} L^{\mu}_{p}(z)U^{\mu}_{p}(t) \\ &\text{when} \\ |z|^{\frac{1}{2}}\sin|\tfrac{1}{2}\arg z| &< |t|^{\frac{1}{2}}\sin|\tfrac{1}{2}\arg t|, \qquad I(z), I(t) \neq 0. \end{aligned}\right\} \tag{4.10}$$

From (4.6), (4.9),

$$B^{\mu}_{n}(z, t) = \mp\frac{i}{\pi} t^{-\mu}e^{t} + (t-z)\sum_{0}^{n}{}_{p} \frac{p!}{\Gamma(p+\mu+1)} L^{\mu}_{p}(z)U^{\mu}_{p}(t). \tag{4.11}$$

From (4.10), (4.11),

$$\left.\begin{aligned} B^{\mu}_{n}(z, t) &= (z-t)\sum_{n+1}^{\infty}{}_{p} \frac{p!}{\Gamma(p+\mu+1)} L^{\mu}_{p}(z)U^{\mu}_{p}(t) \\ &\text{when} \\ |z|^{\frac{1}{2}}\sin|\tfrac{1}{2}\arg z| &< |t|^{\frac{1}{2}}\sin|\tfrac{1}{2}\arg t|, \qquad I(z), I(t) \neq 0. \end{aligned}\right\} \tag{4.12}$$

Likewise, if we define

$$A^{\mu}_{\nu}(z, t) = \frac{\Gamma(\nu+2)}{\Gamma(\mu+\nu+1)}\{L^{\mu}_{\nu+1}(z)L^{\mu}_{\nu}(t) - L^{\mu}_{\nu}(z)L^{\mu}_{\nu+1}(t)\}, \tag{4.13}$$

we have

$$A^{\mu}_{\nu}(z, t) - A^{\mu}_{\nu-1}(z, t) = \frac{\Gamma(\nu+1)}{\Gamma(\mu+\nu+1)}(t-z)L^{\mu}_{\nu}(z)L^{\mu}_{\nu}(t), \tag{4.14}$$

and

$$A^{\mu}_{\nu+n}(z, t) - A^{\mu}_{\nu-1}(z, t) = (t-z)\sum_{0}^{n}{}_{p} \frac{\Gamma(p+\nu+1)}{\Gamma(p+\mu+\nu+1)} L^{\mu}_{\nu+p}(z)L^{\mu}_{\nu+p}(t). \tag{4.15}$$

Since $A^{\mu}_{-1}(z, t) = 0$,

$$A^{\mu}_{n}(z, t) = (t-z)\sum_{0}^{n}{}_{p} \frac{p!}{\Gamma(p+\mu+1)} L^{\mu}_{p}(z)L^{\mu}_{p}(t). \tag{4.16}$$

Now

$$\left(z\frac{\partial}{\partial z} - t\frac{\partial}{\partial t}\right) L^{\mu}_{\nu}(z)U^{\mu}_{\nu}(t) = [\nu L^{\mu}_{\nu} - (\mu+\nu)L^{\mu}_{\nu-1}]U^{\mu}_{\nu} - L^{\mu}_{\nu}[\nu U^{\mu}_{\nu} - (\mu+1)U^{\mu}_{\nu-1}]$$

$$= (\mu+\nu)[L^{\mu}_{\nu}U^{\mu}_{\nu-1} - L^{\mu}_{\nu-1}U^{\mu}_{\nu}] = \frac{\Gamma(\mu+\nu+1)}{\Gamma(\nu+1)} B^{\mu}_{\nu-1}(z, t).$$

Therefore

$$\left.\begin{aligned}\left(z\frac{\partial}{\partial z}-t\frac{\partial}{\partial t}\right)L_\nu^\mu(z)U_\nu^\mu(t)&=\frac{\Gamma(\mu+\nu+1)}{\Gamma(\nu+1)}B_{\nu-1}^\mu(z,t),\\ \text{also,}\qquad\left(z\frac{\partial}{\partial z}-t\frac{\partial}{\partial t}\right)L_\nu^\mu(z)L_\nu^\mu(t)&=\frac{\Gamma(\mu+\nu+1)}{\Gamma(\nu+1)}A_{\nu-1}^\mu(z,t).\end{aligned}\right\}\tag{4.17}$$

$$\begin{aligned}zt\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial t}\right)L_\nu^\mu(z)U_\nu^\mu(t)&=t[-\mu L_\nu^\mu+(\mu+\nu)L_\nu^{\mu-1}]U_\nu^\mu-z[-\mu U_\nu^\mu+(\mu+\nu)U_\nu^{\mu-1}]L_\nu^\mu\\&=\mu(z-t)L_\nu^\mu U_\nu^\mu+(\mu+\nu)\{L_\nu^{\mu-1}[-(\nu+1)U_{\nu+1}^{\mu-1}+(\mu+\nu)U_\nu^{\mu-1}]\\&\qquad-U_\nu^{\mu-1}[-(\nu+1)L_{\nu+1}^{\mu-1}+(\mu+\nu)L_\nu^{\mu-1}]\}\\&=\mu(z-t)L_\nu^\mu U_\nu^\mu+\frac{\Gamma(\mu+\nu+1)}{\Gamma(\nu+1)}B_\nu^{\mu-1}(z,t),\end{aligned}$$

so

$$\left.\begin{aligned}zt\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial t}\right)L_\nu^\mu(z)U_\nu^\mu(t)&=\mu(z-t)L_\nu^\mu(z)U_\nu^\mu(t)+\frac{\Gamma(\mu+\nu+1)}{\Gamma(\nu+1)}B_\nu^{\mu-1}(z,t),\\ \text{and}\qquad zt\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial t}\right)L_\nu^\mu(z)L_\nu^\mu(t)&=\mu(z-t)L_\nu^\mu(z)L_\nu^\mu(t)+\frac{\Gamma(\mu+\nu+1)}{\Gamma(\nu+1)}A_\nu^{\mu-1}(z,t).\end{aligned}\right\}\tag{4.18}$$

$$\begin{aligned}\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial t}\right)L_\nu^\mu(z)U_\nu^\mu(t)&=[-L_\nu^{\mu+1}+L_\nu^\mu]U_\nu^\mu-L_\nu^\mu[-U_\nu^{\mu+1}+U_\nu^\mu]\\&=-L_\nu^{\mu+1}[U_\nu^{\mu+1}-U_{\nu-1}^{\mu+1}]+[L_\nu^{\mu+1}-L_{\nu-1}^{\mu+1}]U_\nu^{\mu+1}=\frac{\Gamma(\mu+\nu+1)}{\Gamma(\nu+1)}B_{\nu-1}^{\mu+1}(z,t),\end{aligned}$$

so

$$\left.\begin{aligned}\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial t}\right)L_\nu^\mu(z)U_\nu^\mu(t)&=\frac{\Gamma(\mu+\nu+1)}{\Gamma(\nu+1)}B_{\nu-1}^{\mu+1}(z,t),\\ \left(\frac{\partial}{\partial z}-\frac{\partial}{\partial t}\right)L_\nu^\mu(z)L_\nu^\mu(t)&=\frac{\Gamma(\mu+\nu+1)}{\Gamma(\nu+1)}A_{\nu-1}^{\mu+1}(z,t).\end{aligned}\right\}\tag{4.19}$$

Let $F_\nu^\mu(z)$, $G_{\nu'}^\mu(z)$ each represent some linear combination of the L and U functions of the orders indicated. By conventional methods we obtain the following indefinite integrals directly from the differential equations:

$$\left.\begin{aligned}&(\nu-\nu')\int e^{-z}z^\mu F_\nu^\mu(z)G_{\nu'}^\mu(z)dz\\&\quad=e^{-z}z^{\mu+1}\left\{F_\nu^\mu(z)\frac{d}{dz}G_{\nu'}^\mu(z)-G_{\nu'}^\mu(z)\frac{d}{dz}F_\nu^\mu(z)\right\}+\text{const.},\end{aligned}\right\}\tag{4.20}$$

$$\left.\begin{aligned}&(\mu+\nu+\nu'+1)\int z^\mu F_\nu^\mu(z)G_{\nu'}^\mu(-z)\,dz\\&\quad=z^{\mu+1}\left\{F_\nu^\mu(z)\frac{d}{d(-z)}G_{\nu'}^\mu(-z)-G_{\nu'}^\mu(-z)\frac{d}{dz}F_\nu^\mu(z)\right\}+\text{const.}\end{aligned}\right\}\tag{4.21}$$

5. *The Functions* $S^\mu_\nu(z)$, $V^\mu_\nu(z)$.

From (1.14), the application of the functions $L^\mu_\nu(z)$, $U^\mu_\nu(z)$ to wave theory will be simplified if we define

$$\left.\begin{aligned} S^\mu_\nu(z) &= z^{\frac{1}{2}\mu}\, e^{-\frac{1}{2}z}\, L^\mu_\nu(z), \\ V^\mu_\nu(z) &= z^{\frac{1}{2}\mu}\, e^{-\frac{1}{2}z}\, U^\mu_\nu(z), \end{aligned}\right\} \tag{5.1}$$

so that

$$\left.\begin{aligned} \psi = \{AS^\mu_\nu(\pm 2ik\xi) + BV^\mu_\nu(\pm 2ik\xi)\}\{CS^\mu_\nu(\mp 2ik\eta) + DV^\mu_\nu(\mp 2ik\eta)\} \\ \times \{Ee^{i\mu\nu} + Fe^{-i\mu\nu}\}e^{-i\omega t} \end{aligned}\right\} \tag{5.2}$$

is a solution to the wave equation.

From (2.46)–(2.60), $S^\mu_\nu(z)$ and $V^\mu_\nu(z)$ satisfy

$$(\mu+\nu)S^\mu_{\nu-1}(z) + (z-\mu-2\nu-1)S^\mu_\nu(z) + (\nu+1)S^\mu_{\nu+1}(z) = 0, \tag{5.3}$$

$$(\mu+\nu)S^{\mu-1}_\nu(z) - (z+\mu)z^{-1}S^\mu_\nu(z) + S^{\mu+1}_\nu(z) = 0, \tag{5.4}$$

$$z^{\frac{1}{2}}S^\mu_\nu(z) = -(\nu+1)S^{\mu-1}_{\nu+1}(z) + (\mu+\nu)S^{\mu-1}_\nu(z), \tag{5.5}$$

$$z^{\frac{1}{2}}S^\mu_\nu(z) = (z-\nu)S^{\mu-1}_\nu(z) + (\mu+\nu-1)S^{\mu-1}_{\nu-1}(z), \tag{5.6}$$

$$(\mu+\nu+1)z^{\frac{1}{2}}S^\mu_\nu(z) = (\nu+1)S^{\mu+1}_{\nu+1}(z) + (z-\nu-1)S^{\mu+1}_\nu(z), \tag{5.7}$$

$$z^{\frac{1}{2}}S^\mu_\nu(z) = S^{\mu+1}_\nu(z) - S^{\mu+1}_{\nu-1}(z), \tag{5.8}$$

$$\nu S^\mu_\nu(z) = -z^{\frac{1}{2}}S^{\mu+1}_{\nu-1}(z) + (\mu+\nu)S^\mu_{\nu-1}(z), \tag{5.9}$$

$$\nu S^\mu_\nu(z) = (\nu-z)S^\mu_{\nu-1}(z) + (\mu+\nu-1)z^{\frac{1}{2}}S^{\mu-1}_{\nu-1}(z), \tag{5.10}$$

$$(\mu+\nu+1)S^\mu_\nu(z) = z^{\frac{1}{2}}S^{\mu+1}_{\nu+1}(z) + (\nu+1-z)S^\mu_{\nu+1}(z), \tag{5.11}$$

$$S^\mu_\nu(z) = S^\mu_{\nu-1}(z) - z^{\frac{1}{2}}S^{\mu-1}_{\nu-1}(z), \tag{5.12}$$

$$z\frac{d}{dz}S^\mu_\nu(z) = (\nu+1)S^\mu_{\nu+1}(z) + (\tfrac{1}{2}z - \tfrac{1}{2}\mu - \nu - 1)S^\mu_\nu(z), \tag{5.13}$$

$$z\frac{d}{dz}S^\mu_\nu(z) = (\tfrac{1}{2}\mu + \nu - \tfrac{1}{2}z)S^\mu_\nu(z) - (\mu+\nu)S^\mu_{\nu-1}(z), \tag{5.14}$$

$$z\frac{d}{dz}S^\mu_\nu(z) = -z^{\frac{1}{2}}S^{\mu+1}_\nu(z) + (\tfrac{1}{2}z + \tfrac{1}{2}\mu)S^\mu_\nu(z), \tag{5.15}$$

$$z\frac{d}{dz}S^\mu_\nu(z) = -(\tfrac{1}{2}z + \tfrac{1}{2}\mu)S^\mu_\nu(z) + (\mu+\nu)z^{\frac{1}{2}}S^{\mu-1}_\nu(z). \tag{5.16}$$

From (2.15),(3.22),

$$\lim_{s\to 0}\left(\frac{s}{\nu}\right)^{\frac{1}{2}\mu} S^\mu_{\nu/s}(sz) = J_\mu(2\sqrt{\nu z}), \tag{5.17}$$

$$\left.\begin{aligned} \lim_{s\to 0}\left(\frac{s}{\nu}\right)^{\frac{1}{2}\mu} V^\mu_{\nu/s}(sz) &= H^{(1)}_\mu(2\sqrt{\nu z}), \qquad 0 < \arg z < \pi, \\ &= H^{(2)}_\mu(2\sqrt{\nu z}), \qquad -\pi < \arg z < 0. \end{aligned}\right\} \tag{5.18}$$

From (2.1), (3.16), (3.17),

$$S_n^m(z) \simeq \frac{(m+n)!}{m!n!} z^{\frac{1}{2}m}, \qquad m, n \geq 0, \qquad |z| \ll 1, \tag{5.19}$$

$$\left.\begin{aligned} V_n^m(z) &\simeq \mp \frac{i}{\pi}(m-1)! z^{-\frac{1}{2}m}, \quad m > 0, \qquad n \geq -m, \qquad |z| \ll 1, \\ V_n^0(z) &\simeq 1 \pm \frac{i}{\pi}\left(\ln z + \gamma + \sum_1^n{}_k \frac{1}{k}\right), \qquad n \geq 0, \qquad |z| \ll 1. \end{aligned}\right\} \tag{5.20}$$

From (2.14), (3.20),

$$S_\nu^\mu(\pm ix) \sim \frac{e^{\pm\frac{1}{2}\pi i(\frac{1}{2}\mu+\nu)}}{\Gamma(\nu+1)} x^{\frac{1}{2}\mu+\nu} e^{\mp\frac{1}{2}ix}, \qquad R(\mu+2\nu) > -1, \quad x >> (\nu+1)(\mu+\nu+1), \tag{5.21}$$

$$V_\nu^\mu(\pm ix) \sim \frac{1}{\pi}\Gamma(\mu+\nu+1) e^{\pm\frac{1}{2}\pi i(-\frac{1}{2}\mu+\nu)} x^{-\frac{1}{2}\mu-\nu-1} e^{\pm\frac{1}{2}ix}, \quad x >> (\nu+1)(\mu+\nu+1), \tag{5.22}$$

where in (5.21) and (5.22), of course, our double sign convention does not apply.

From (2.64), (3.24),

$$S_\nu^\mu(z) = \frac{1}{\sqrt{\pi}} \nu^{\frac{1}{2}\mu-\frac{1}{4}} z^{-\frac{1}{4}} \cos(2\sqrt{\nu z} - \tfrac{1}{2}\pi\mu - \tfrac{1}{4}\pi)[1 + 0(\nu^{-\frac{1}{2}})], \tag{5.23}$$

$$V_\nu^\mu(z) = \frac{1}{\sqrt{\pi}} \nu^{\frac{1}{2}\mu-\frac{1}{4}} z^{-\frac{1}{4}} e^{\pm i(2\sqrt{\nu z}-\frac{1}{2}\pi\mu-\frac{1}{4}\pi)}[1 + 0(\nu^{-\frac{1}{2}})]. \tag{5.24}$$

From (4.10),

$$\left.\begin{aligned} &\pm \frac{i}{\pi}\left(\frac{z}{t}\right)^{\frac{1}{2}\mu} \frac{e^{\frac{1}{2}(t-z)}}{t-z} = \sum_0^\infty{}_p \frac{p!}{\Gamma(p+\mu+1)} S_p^\mu(z) V_p^\mu(t), \\ &\text{when} \\ &|z|^{\frac{1}{2}} \sin|\tfrac{1}{2}\arg z| < |t|^{\frac{1}{2}} \sin|\tfrac{1}{2}\arg t|, \qquad I(z), I(t) \neq 0, \end{aligned}\right\} \tag{5.25}$$

where the double sign pertains to t.

Now let

$$\left.\begin{aligned} C_\nu^\mu(z,t) &= (zt)^{\frac{1}{2}\mu} e^{-\frac{1}{2}(z+t)} A_\nu^\mu(z,t) \\ &= \frac{\Gamma(\nu+2)}{\Gamma(\mu+\nu+1)} \{S_{\nu+1}^\mu(z) S_\nu^\mu(t) - S_\nu^\mu(z) S_{\nu+1}^\mu(t)\} \\ D_\nu^\mu(z,t) &= (zt)^{\frac{1}{2}\mu} e^{-\frac{1}{2}(z+t)} B_\nu^\mu(z,t) \\ &= \frac{\Gamma(\nu+2)}{\Gamma(\mu+\nu+1)} \{S_{\nu+1}^\mu(z) V_\nu^\mu(t) - S_\nu^\mu(z) V_{\nu+1}^\mu(t)\}. \end{aligned}\right\} \tag{5.26}$$

From (4.12),

$$\left.\begin{aligned} D^{\mu}_{n}(z,t) &= (z-t)\sum_{n+1}^{\infty}{}^{p}\frac{p!}{\Gamma(p+\mu+1)}\,S^{\mu}_{p}(z)V^{\mu}_{p}(t), \\ &\text{when} \\ &|z|^{\frac{1}{2}}\sin|\tfrac{1}{2}\arg z| < |t|^{\frac{1}{2}}\sin|\tfrac{1}{2}\arg t|, \qquad I(z), I(t) \neq 0. \end{aligned}\right\} \tag{5.27}$$

From (4.16),

$$C^{\mu}_{n}(z,t) = (t-z)\sum_{0}^{n}{}^{p}\frac{p!}{\Gamma(p+\mu+1)}\,S^{\mu}_{p}(z)S^{\mu}_{p}(t). \tag{5.28}$$

From (4.17),

$$\left.\begin{aligned} \left(z\frac{\partial}{\partial z}-t\frac{\partial}{\partial t}\right)S^{\mu}_{\nu}(z)V^{\mu}_{\nu}(t) &= \frac{\Gamma(\mu+\nu+1)}{\Gamma(\nu+1)}\,D^{\mu}_{\nu-1}(z,t) \\ &\quad -\tfrac{1}{2}(z-t)S^{\mu}_{\nu}(z)V^{\mu}_{\nu}(t), \\ \left(z\frac{\partial}{\partial z}-t\frac{\partial}{\partial t}\right)S^{\mu}_{\nu}(z)S^{\mu}_{\nu}(t) &= \frac{\Gamma(\mu+\nu+1)}{\Gamma(\nu+1)}\,C^{\mu}_{\nu-1}(z,t) \\ &\quad -\tfrac{1}{2}(z-t)S^{\mu}_{\nu}(z)S^{\mu}_{\nu}(t). \end{aligned}\right\} \tag{5.29}$$

From (4.18),

$$\left.\begin{aligned} zt\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial t}\right)S^{\mu}_{\nu}(z)V^{\mu}_{\nu}(t) &= \frac{\Gamma(\mu+\nu+1)}{\Gamma(\nu+1)}\,(zt)^{\frac{1}{2}}\,D^{\mu-1}_{\nu}(z,t) \\ &\quad +\tfrac{1}{2}\mu(z-t)S^{\mu}_{\nu}(z)V^{\mu}_{\nu}(t), \\ zt\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial t}\right)S^{\mu}_{\nu}(z)S^{\mu}_{\nu}(t) &= \frac{\Gamma(\mu+\nu+1)}{\Gamma(\nu+1)}\,(zt)^{\frac{1}{2}}\,C^{\mu-1}_{\nu}(z,t) \\ &\quad +\tfrac{1}{2}\mu(z-t)S^{\mu}_{\nu}(z)S^{\mu}_{\nu}(t). \end{aligned}\right\} \tag{5.30}$$

From (4.19),

$$\left.\begin{aligned} \left(\frac{\partial}{\partial z}-\frac{\partial}{\partial t}\right)S^{\mu}_{\nu}(z)V^{\mu}_{\nu}(t) &= \frac{\Gamma(\mu+\nu+1)}{\Gamma(\nu+1)}\,(zt)^{-\frac{1}{2}}\,D^{\mu+1}_{\nu-1}(z,t) \\ &\quad -\tfrac{1}{2}\mu\,\frac{z-t}{zt}\,S^{\mu}_{\nu}(z)V^{\mu}_{\nu}(t), \\ \left(\frac{\partial}{\partial z}-\frac{\partial}{\partial t}\right)S^{\mu}_{\nu}(z)S^{\mu}_{\nu}(t) &= \frac{\Gamma(\mu+\nu+1)}{\Gamma(\nu+1)}\,(zt)^{-\frac{1}{2}}\,C^{\mu+1}_{\nu-1}(z,t) \\ &\quad -\tfrac{1}{2}\mu\,\frac{z-t}{zt}\,S^{\mu}_{\nu}(z)S^{\mu}_{\nu}(t). \end{aligned}\right\} \tag{5.31}$$

Now $z = -\xi + \eta$, $r = \xi + \eta$. Therefore in (5.2), by (5.21), (5.22),

$$\left.\begin{array}{ll} S_\nu^\mu(-2ik\xi)V_\nu^\mu(+2ik\eta) & \text{represents a wave traveling in the outward direction} \\ V_\nu^\mu(2ik\xi)S_\nu^\mu(-2ik\eta) & \text{" " " " " " outward "} \\ S_\nu^\mu(2ik\xi)V_\nu^\mu(-2ik\eta) & \text{" " " " " " inward "} \\ V_\nu^\mu(-2ik\xi)S_\nu^\mu(2ik\eta) & \text{" " " " " " inward "} \end{array}\right\} \quad (5.32)$$

$$\left.\begin{array}{ll} S_\nu^\mu(2ik\xi)S_\nu^\mu(-2ik\eta) & \text{" " " " " " } +z \text{ "} \\ V_\nu^\mu(-2ik\xi)V_\nu^\mu(2ik\eta) & \text{" " " " " " } +z \text{ "} \\ S_\nu^\mu(-2ik\xi)S_\nu^\mu(+2ik\eta) & \text{" " " " " " } -z \text{ "} \\ V_\nu^\mu(2ik\xi)V_\nu^\mu(-2ik\eta) & \text{" " " " " " } -z \text{ "} \end{array}\right\} \quad (5.33)$$

From (4.2), (4.3),

$$V_\nu^\mu(z)\frac{d}{dz}S_\nu^\mu(z) - S_\nu^\mu(z)\frac{d}{dz}V_\nu^\mu(z) = \mp\frac{i}{\pi}\frac{\Gamma(\mu+\nu+1)}{\Gamma(\nu+1)}\frac{1}{z}, \qquad R(\mu) > -1, \quad (5.34)$$

$$V_\nu^\mu(z)S_{\nu+1}^\mu(z) - S_\nu^\mu(z)V_{\nu+1}^\mu(z) = \mp\frac{i}{\pi}\frac{\Gamma(\mu+\nu+1)}{\Gamma(\nu+1)}, \qquad R(\mu) > -1. \quad (5.35)$$

From (4.20), (4.21), if $H_\nu^\mu(z)$, $I_\nu^\mu(z)$ each represent some linear combination of the S and V functions of the order indicated, the following indefinite integrals hold:

$$(\nu - \nu')\int H_\nu^\mu(z)I_{\nu'}^\mu(z)\,dz = z\left\{H_\nu^\mu(z)\frac{d}{dz}I_{\nu'}^\mu(z) - I_{\nu'}^\mu(z)\frac{d}{dz}H_\nu^\mu(z)\right\} + \text{const.}, \quad (5.36)$$

$$(\mu+\nu+\nu'+1)\int H_\nu^\mu(z)I_{\nu'}^\mu(-z)\,dz = z\left\{H_\nu^\mu(z)\frac{d}{d(-z)}I_{\nu'}^\mu(-z) - I_{\nu'}^\mu(-z)\frac{d}{dz}H_\nu^\mu(z)\right\} + \text{const.} \quad (5.37)$$

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STRESSES AND SMALL DISPLACEMENTS OF SHALLOW SPHERICAL SHELLS. I

By Eric Reissner

Introduction. The purpose of the present paper is to derive a system of equations which can be used for the analysis of shallow segments of thin, elastic, spherical shells. A segment will be called shallow if the ratio of its height to base diameter is less than, say, $\frac{1}{5}$. The results obtained on the basis of this assumption will often also be applicable to shells which are not shallow, namely then, when the loads are such that the stresses are effectively restricted to shallow zones.

The problem of the spherical elastic shell has been the subject of numerous researches. For the rotationally symmetric case the fundamental results were obtained in 1912 (1) and have been the starting point of many applications. While it is possible to deduce from these results approximate equations equivalent to part of what follows, it is believed that the present approach to the problem of the *shallow* shell may be of some interest even for rotationally symmetric cases.

A number of investigations have been concerned with the shell loaded in a non-rotationally symmetric manner (2, 3, 4). In its general form this problem is quite difficult and the results so far obtained are not easy to apply. Restricting attention to the shallow shell in the manner of the present paper brings with it a very considerable simplification of the analysis.

Formulation of the problem. Let the equation of the middle surface of the shell be given in the form

$$z = \sqrt{R^2 - r^2} - (R - h) \tag{1}$$

where R is the radius of the shell, $z(r)$ the distance from base to middle surface, and r the distance from the apex of the shell measured in a plane parallel to the base plane.

The assumption of shallowness is expressed by the following order of magnitude relation

$$\frac{dz}{dr} = -\frac{r}{\sqrt{R^2 - r^2}} \approx -\frac{r}{R} = o(1) \tag{2}$$

the symbol $o(1)$ indicating that for the significant values of r the value of the ratio r/R is small compared with unity.

Let N_{rr}, $N_{\theta\theta}$, $N_{r\theta}$ designate the direct stress resultants in meridional and circumferential direction and V_r and V_θ the transverse shear stress resultants. Let M_{rr}, $M_{\theta\theta}$, and $M_{r\theta}$ designate the stress couples, in meridional and circumferential direction. Let p_r, p_θ and p designate the components of load intensity in meridional circumferential and normal direction, respectively.

It can then be shown that the following differential equations of equilibrium hold

$$\frac{\partial r N_{rr}}{\partial r} + \frac{\partial N_{r\theta}}{\partial \theta} - N_{\theta\theta} + \frac{r}{R} V_r + r p_r = 0 \tag{3a}$$

$$\frac{\partial r N_{r\theta}}{\partial r} + \frac{\partial N_{\theta\theta}}{\partial \theta} + N_{r\theta} + \frac{r}{R} V_\theta + r p_\theta = 0 \tag{3b}$$

$$\frac{\partial r V_r}{\partial r} + \frac{\partial V_\theta}{\partial \theta} - \frac{r}{R}(N_{rr} + N_{\theta\theta}) + r p = 0 \tag{4a}$$

$$\frac{\partial r M_{rr}}{\partial r} + \frac{\partial M_{r\theta}}{\partial \theta} - M_{\theta\theta} - r V_r = 0 \tag{4b}$$

$$\frac{\partial r M_{r\theta}}{\partial r} + \frac{\partial M_{\theta\theta}}{\partial \theta} + M_{r\theta} - r V_\theta = 0 \tag{4c}$$

The stress strain relations are written in the following form

$$\epsilon_r = \frac{1}{tE}(N_{rr} - \nu N_{\theta\theta}) \tag{5a}$$

$$\epsilon_\theta = \frac{1}{tE}(N_{\theta\theta} - \nu N_{rr}) \tag{5b}$$

$$\gamma_{r\theta} = \frac{2(1+\nu)}{tE} N_{r\theta} \tag{5c}$$

$$M_{rr} = D(\kappa_r + \nu \kappa_\theta) \tag{6a}$$

$$M_{\theta\theta} = D(\kappa_\theta + \nu \kappa_r) \tag{6b}$$

$$M_{r\theta} = \frac{1-\nu}{2} D\tau_{r\theta} \tag{6c}$$

In these equations t stands for the thickness of the shell and $D = Et^3/12(1-\nu^2)$. The components of direct and bending strain are taken in terms of meridional, circumferential and normal components of displacement u, v and w, in the form

$$\epsilon_r = \frac{\partial u}{\partial r} + \frac{w}{R} \tag{7a}$$

$$\epsilon_\theta = \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r} + \frac{w}{R} \tag{7b}$$

$$\gamma_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + r\frac{\partial}{\partial r}\left(\frac{v}{r}\right) \tag{7c}$$

$$\kappa_r = -\frac{\partial^2 w}{\partial r^2} \tag{8a}$$

$$\kappa_\theta = -\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} - \frac{1}{r}\frac{\partial w}{\partial r} \tag{8b}$$

$$\tau_{r\theta} = -2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right) \tag{8c}$$

It may be noted that eqs. (8) are the same expressions as those in the theory of bending of plates. Use of such expressions for the analysis of shallow shells is supported by known results for the buckling of circular cylindrical shells where the buckling modes subdivide the shell into independent shallow panels.

Eqs. (3) to (8) may be written as a system of eleven equations for eleven unknowns, eight of them stress resultants and couples and three of them displacements. The problem consists in reducing these equations to a form which is convenient for the purpose of obtaining explicit solutions.

Reduction of the problem to two simultaneous equations for two functions. The two simultaneous equations are for a stress function F and for the component of displacement w. Their use is suggested by the theory of plane stress and the theory of plate bending to which the present formulation reduces when $R = \infty$.

The observation is made that for shallow shells the effect of the transverse shear terms in the equations (3) of force equilibrium in meridional and circumferential direction will be negligible. Proceeding on this assumption—which may be justified by substituting in the end in equations (3) the transverse shears obtained, and by comparing their order of magnitude with the order of magnitude of the terms which were retained—equations (3) reduce to the equilibrium equations of plane stress. As in the theory of plane stress they may be satisfied by means of a stress function F.

Restricting attention to the case where the external load terms p_r and p_θ are derivable from a load potential Ω,

$$p_r = -\frac{\partial \Omega}{\partial r}, \qquad p_\theta = -\frac{1}{r}\frac{\partial \Omega}{\partial \theta} \tag{9a, b}$$

the simplified equations (3) are satisfied by setting

$$N_{\theta\theta} = \frac{\partial^2 F}{\partial r^2} + \Omega \tag{10a}$$

$$N_{rr} = \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2} + \Omega \tag{10b}$$

$$N_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F}{\partial \theta}\right) \tag{10c}$$

As in the theory of plane stress a differential equation for F is obtained from the relevant equation of compatibility. It may be verified that equations (7) correspond to a compatibility relation of the form

$$\frac{1}{r^2}\frac{\partial^2 \epsilon_r}{\partial \theta^2} - \frac{1}{r}\frac{\partial \epsilon_r}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \epsilon_\theta}{\partial r}\right) - \frac{1}{r^2}\frac{\partial^2 (r\gamma_{r\theta})}{\partial r \partial \theta} = \frac{1}{R}\nabla^2 w \tag{11}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \tag{12}$$

Deduction of equation (11) was facilitated by the fact that its form in the case $R = \infty$ could be found in the literature on the subject (5).

Combining equations (10) and (5) there follows from equation (11)—without actually carrying out most of the calculations as the result for $R = \infty$ is known and more easily established in cartesian coordinates—as first fundamental equation.

$$\boxed{\nabla^2 \nabla^2 F - \frac{tE}{R} \nabla^2 w = -(1 - \nu)\nabla^2 \Omega} \tag{13}$$

To obtain the second differential equation involving F and w the example of plate bending theory is followed. The moment equilibrium equations (4b) and (4c) are introduced into the transverse force equilibrium equation (4a), giving

$$\frac{\partial}{\partial r}\left(\frac{\partial r M_{rr}}{\partial r} + \frac{\partial M_{r\theta}}{\partial \theta} - M_{\theta\theta}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial r M_{r\theta}}{\partial r} + \frac{\partial M_{\theta\theta}}{\partial \theta} + M_{r\theta}\right) - \frac{r}{R}(N_{rr} + N_{\theta\theta}) + rp = 0 \tag{14}$$

Expressing the stress couples in terms of the deflection w,

$$M_{rr} = -D\left[\frac{\partial^2 w}{\partial r^2} + \nu\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\right] \tag{15a}$$

$$M_{\theta\theta} = -D\left[\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \nu\frac{\partial^2 w}{\partial r^2}\right] \tag{15b}$$

$$M_{r\theta} = -(1 - \nu)D\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right) \tag{15c}$$

and substituting the result in equation (14) there follows,

$$-D\nabla^2 \nabla^2 w - \frac{1}{R}(N_{rr} + N_{\theta\theta}) + p = 0. \tag{16}$$

Taking into account equations (10) there is obtained the second fundamental equation

$$\boxed{D\nabla^2 \nabla^2 w + \frac{1}{R}\nabla^2 F = p - \frac{2\Omega}{R}} \tag{17}$$

Having F and w the direct stress resultants and the stress couples follow, according to equations (10) and (15), by differentiation. From equations

(4b, c) and (15) follows for the transverse shear stress resultants, as in plate theory,

$$V_r = -D\frac{\partial \nabla^2 w}{\partial r}, \qquad V_\theta = -D\frac{1}{r}\frac{\partial \nabla^2 w}{\partial \theta} \tag{18a, b}$$

For the satisfaction of boundary conditions it is to be noted that within the approximation of the present theory "horizontal" force and "vertical" displacement are given by N_{rr}, $N_{r\theta}$ and w. For vertical force V_r and horizontal displacement u_h the following expressions must be used

$$V_r = V_r - \frac{r}{R}N_{rr} \tag{19}$$

$$u_h = u + \frac{r}{R}w \tag{20}$$

Again, as in plate theory, the vertical edge reaction R_r contains the edgewise rate of change of twisting couple

$$R_r = V_r + \frac{1}{r}\frac{\partial M_{r\theta}}{\partial \theta} \tag{21}$$

It is apparent that solutions of equations (13) and (17) may be written in the form

$$F(r, \theta) = \sum \cos n\theta \, f_n(r) \tag{22a}$$

$$w(r, \theta) = \sum \cos n\theta \, g_n(r) \tag{22b}$$

and that except for the particular integrals the functions f_n and g_n are combinations of elementary functions and of Bessel functions with argument proportional to $\sqrt{i}$.

In a sequel to the present paper the solution will be evaluated for rotationally symmetric conditions and the results applied to the problem of the shell with normal load applied to a small but finite area.

A forthcoming dissertation (6) deals with the general problem without rotational symmetry, and also gives additional applications of the solution with rotational symmetry.

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NOTE ON COEFFICIENTS FOR NUMERICAL INTEGRATION WITH DIFFERENCES

By Herbert E. Salzer[1]

In a recent paper,[2] quantities $B_i^{(n)}(p)$ are tabulated to 10 decimal places, for continuous numerical integration (i.e. integration to various points within an interval of tabulation) by a Lagrangian formula which uses the tabular entries only. The present note indicates how those same quantities $B_i^{(n)}(p)$ can be employed as they stand, for continuous numerical integration using differences, in formulas obtained by integrating the interpolation formulas of Gregory-Newton, Newton-Gauss (two forms), Everett, and Steffensen. The customary Δ and δ notation for advancing and central differences is utilized (e.g. $\delta_1^5 = \Delta_{-2}^5 = \Delta^5 f(x_0 - 2h)$), and h denotes the tabular interval.

A. Gregory-Newton (Advancing differences)

$$\int_{x_0}^{x_0+ph} f(x)\,dx \sim h\left[p\,f(x_0) + \frac{p^2}{2}\Delta_0 + \sum_{i=2}^{M}\int_0^p \binom{t}{i} dt\,\Delta_0^i\right]$$

Coefficient of $\Delta_0^2 = B_{-1}^{(3)}(p)$, tabulated for $p = -1(.01)1$.

" " $\Delta_0^3 = -B_{-1}^{(4)}(p)$ " " $p = -1(.01)2$.

" " $\Delta_0^4 = [B_2^{(5)}(1) - B_2^{(5)}(1-p)]$, latter term tabulated for $p = -1(.01)3$.

" " $\Delta_0^5 = [B_3^{(6)}(2-p) - B_3^{(6)}(2)]$, former term tabulated for $p = -1(.1)4$.

" " $\Delta_0^6 = [B_3^{(7)}(2) - B_3^{(7)}(2-p)]$, latter term tabulated for $p = -1(.1)5$.

Thus the integrated Gregory-Newton formula can be used for $p = -1$ to 1, at intervals of 0.01 through Δ_0^4, and at intervals of 0.1 through Δ_0^6.

B. Newton-Gauss (Forward formula)

$$\int_{x_0}^{x_0+ph} f(x)\,dx \sim h\left[p\,f(x_0) + \frac{p^2}{2}\Delta_0 + \sum_{i=1}^{M}\left(\int_0^p \binom{t+i-1}{2i} dt\,\delta_0^{2i} + \int_0^p \binom{t+i}{2i+1} dt\,\delta_1^{2i+1}\right)\right]$$

Coefficient of $\delta_0^2 = B_{-1}^{(3)}(p)$, (See A.)

" " $\delta_1^3 = B_2^{(4)}(p)$, tabulated for $p = -1(.01)2$.

" " $\delta_0^4 = B_{-2}^{(5)}(p)$, " " $p = -2(.01)2$.

" " $\delta_1^5 = B_3^{(6)}(p)$, " " $p = -2(.1)3$.

" " $\delta_0^6 = B_{-3}^{(7)}(p)$, " " $p = -3(.1)3$.

[1] Mathematical Tables Project, National Bureau of Standards, New York City.

[2] A. N. Lowan, H. E. Salzer "Table of Coefficients for Numerical Integration Without Differences," Jour. of Math. & Phys. vol. xxiv No. 1, Feb. 1945, pp. 1-21.

Thus the integrated Newton-Gauss forward formula can be used for $p = -1$ to 1, at intervals of 0.01 through δ_0^4, and at intervals of 0.1 through δ_0^6.

C. Newton-Gauss ("Backward formula," but rewritten with an obvious shift in argument so that x_0 became $x_0 + h$ or x_1.)

$$\int_{x_0}^{x_0+ph} f(x)dx \sim h\left[pf(x_0) + \frac{p^2}{2}\Delta_0 + \sum_{i=1}^{M}\left(\int_0^p \binom{t+i-1}{2i} dt\, \delta_1^{2i} + \int_0^p \binom{t+i-1}{2i+1} dt\, \delta_1^{2i+1}\right)\right]^3$$

Coefficient of $\delta_1^2 = B_{-1}^{(3)}(p)$, (See A.)

" " $\delta_1^3 = -B_{-1}^{(4)}(p)$, (See A.)

" " $\delta_1^4 = B_{-2}^{(5)}(p)$, (See B.)

" " $\delta_1^5 = -B_{-2}^{(6)}(p)$, tabulated for $p = -2(.1)3$.

" " $\delta_1^6 = B_{-3}^{(7)}(p)$, (See B.)

Thus this variation of the integrated Newton-Gauss "backward formula" can be used for $p = -1$ to 1 at intervals of 0.01 through δ_1^4 and at intervals of 0.1 through δ_1^6. By using $B_{-1}^{(3)}(1) - B_{-1}^{(3)}(1-p)$ in place of $B_{-1}^{(3)}(p)$, this formula can be used up to $p = 2$.

D. Everett formula (Even-order central differences)

$$\int_{x_0}^{x_0+ph} f(x)dx \sim h\left[pf(x_0) + \frac{p^2}{2}\Delta_0 + \sum_{i=1}^{M}\left(\int_0^p \binom{t+i}{2i+1} dt\, \delta_1^{2i} + \int_0^p \binom{-t+i+1}{2i+1} dt\, \delta_0^{2i}\right)\right]$$

Coefficient of $\delta_1^2 = B_2^{(4)}(p)$, (See B.)

" " $\delta_0^2 = B_{-1}^{(4)}(p)$, (See A.)

" " $\delta_1^4 = B_3^{(6)}(p)$, (See B.)

" " $\delta_0^4 = B_{-2}^{(6)}(p)$, (See C.)

Thus the integrated Everett formula can be used for $p = -1$ to 2 at intervals of 0.01 through δ^2 (this takes account of 3rd differences) and at intervals of 0.1 through δ^4 (this takes account of 5th differences). Hence in one sense the table of $B_i^{(n)}(p)$ is not as well adapted for the integrated Everett formula as it is for the other formulas, since it does not provide 4th difference coefficients at intervals of 0.01. On the other hand, it can be used up to $p = 2$, (but see C.).

E. Steffensen formula (Odd-order central differences)

$$\int_{x_0}^{x_0+ph} f(x)dx \sim h\left[pf(x_0) + \sum_{i=1}^{M}\left(\int_0^p \binom{t+i}{2i} dt\, \delta_1^{2i-1} + \int_0^p -\binom{t+i-1}{2i} dt\, \delta_{-1}^{2i-1}\right)\right]$$

[3] Note that $\delta_1^{2i+1} = \Delta_{-i}^{2i+1}$.

Coefficient of $\delta_1 = B_1^{(3)}(p)$, tabulated for $p = -1(.01)1$.

" " $\delta_{-1} = -B_{-1}^{(3)}(p)$, (See A.)

" " $\delta_1^3 = B_2^{(5)}(p)$, tabulated for $p = -2(.01)2$.

" " $\delta_{-1}^3 = -B_{-2}^{(5)}(p)$, (See B.)

" " $\delta_1^5 = B_3^{(7)}(p)$, tabulated for $p = -3(.1)3$.

" " $\delta_{-1}^5 = -B_{-3}^{(7)}(p)$, (See B.)

Thus the integrated Steffensen formula can be used for $p = -1$ to 1, at intervals 0.01 through δ^3 (this takes account of 4th differences), and at intervals of 0.1 through δ^5 (this takes account of 6th differences).

All of the above relations between the coefficients of the differences and $B_i^{(n)}(p)$ follow readily from the relation between binomial coefficients and Lagrange interpolation coefficients $A_i^{(n)}(p)$, where $B_i^{(n)}(p) = \int_0^p A_i^{(n)}(t)dt$. Use is made of the identities $\binom{p}{i} = (-1)^i A_{[(i+1)/2]}^{(i+1)}$ ($[(i-1)/2] - p$), (where $[x]$ here denotes the largest integer in x), $A_i^{(n)}(p) = A_{-i}^{(n)}(-p)$ for n odd, $A_i^{(n)}(p) = A_{1-i}^{(n)}(1-p)$ for n even, $A_{-i}^{(2i+1)}(p+1) = A_i^{(2i+1)}(p)$, and $A_i^{(2i)}(p) = -A_i^{(2i)}(-p)$.

REDUCIBLE DYNAMICAL SYSTEMS

By T. Y. Thomas

Consider a conservative dynamical system D referred to generalized coordinates $x^1, \cdots, x^n$. Assume the kinetic energy T and potential function V to be of the form

$$T = \tfrac{1}{2} g_{\alpha\beta}(x) \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt}, \qquad V = V(x^1, \cdots, x^n).$$

Suppose it is possible to effect a coordinate transformation $x \leftrightarrow y$ such that, relative to the y coordinates, we have $T = T_1 + T_2$ and $V = V_1 + V_2$ where

$$T_1 = \tfrac{1}{2} \sum_1^r a_{\alpha\beta}(y^1, \cdots, y^r) \frac{dy^\alpha}{dt} \frac{dy^\beta}{dt}, \qquad V_1 = V_1(y^1, \cdots, y^r),$$

$$T_2 = \tfrac{1}{2} \sum_{r+1}^n b_{\alpha\beta}(y^{r+1}, \cdots, y^n) \frac{dy^\alpha}{dt} \frac{dy^\beta}{dt}, \qquad V_2 = V_2(y^{r+1}, \cdots, y^n),$$

with $0 < r < n$. The system can then be regarded as the combination of two separate systems D_1 and D_2 having kinetic energies and potential functions given by T_1, V_1 and V_2, V_2 respectively. Such a system D will be said to be *reducible*. We now prove the following result.

The system D will be reducible if, and only if, it admits a quadratic first integral

$$\tfrac{1}{2} h_{\alpha\beta}(x) \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} + W = \text{const.}, \tag{1}$$

distinct from the energy integral[1], *such that* $h_{\alpha\beta,\gamma} = 0$ *where the quantities* $h_{\alpha\beta,\gamma}$ *are the components of the covariant derivative*[2] *of the tensor h defined by the above coeficients* $h_{\alpha\beta}$.

Suppose first that D is reducible so that the above coordinates y can be introduced. Denote by $\Lambda^\alpha_{\beta\gamma}$ the Christoffel symbols based on the $g_{\alpha\beta}$ when taken relative to the y coordinates. Then $\Lambda^\alpha_{\beta\gamma} = A^\alpha_{\beta\gamma}(\alpha, \beta, \gamma = 1, \cdots r)$ and $\Lambda^\alpha_{\beta\gamma} = B^\alpha_{\beta\gamma}(\alpha, \beta, \gamma = r+1, \cdots, n)$ where the A's and the B's are the Christoffel symbols based on the components $a_{\alpha\beta}$ and $b_{\alpha\beta}$ respectively. Other components $\Lambda^\alpha_{\beta\gamma}$ are

[1] The statement that (1) is distinct from the energy integral

$$\tfrac{1}{2} g_{\alpha\beta} \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} + V = \text{const.},$$

means precisely that constants c and k cannot be found such that $h_{\alpha\beta} = c g_{\alpha\beta}$ and $W = cV + k$; in fact the non-existence of a constant c such that $h_{\alpha\beta} = c g_{\alpha\beta}$ suffices to meet this requirement. The energy integral and the integral (1) can then be taken as members of the set of linearly independent integrals in the basis of integrals of this type. See, T. Y. Thomas, *The fundamental theorem on quadratic first integrals*, Proc. N.A.S., vol. 32, 1946, pp. 10–15.

[2] We understand here, and in the following, that covariant differentiation is based on the tensor components $g_{\alpha\beta}$ which determine the kinetic energy of the given dynamical system.

zero identically. Hence if we define the quantities $k_{\alpha\beta}(\alpha, \beta = 1, \cdots, n)$ by the matrix

$$\left\| \begin{array}{c|c} \begin{matrix} a_{11} & \cdots & a_{1r} \\ \cdots & \cdots & \cdots \\ a_{r1} & \cdots & a_{rr} \end{matrix} & \begin{matrix} 0 \\ \cdots \\ 0 \end{matrix} \\ \hline \begin{matrix} 0 & \cdots & 0 \end{matrix} & 0 \end{array} \right\|$$

we readily see that

$$\tfrac{1}{2} k_{\alpha\beta} \frac{dy^\alpha}{dt} \frac{dy^\beta}{dt} + V_1(y^1, \cdots, y^r) = \text{const.},$$

is an integral of D and also that $k_{\alpha\beta,\gamma} = 0$. This integral is distinct from the energy integral. Transforming to the x coordinates we obtain (1) with $h_{\alpha\beta,\gamma} = 0$.

Conversely assume the existence of an integral (1), distinct from the energy integral, and for which $h_{\alpha\beta,\gamma} = 0$. There is no loss in generality in supposing that det. $|h_{\alpha\beta}| = 0$ at a point P. For if det. $|h_{\alpha\beta}| \neq 0$ consider the determinant equation $|g_{\alpha\beta} - \sigma h_{\alpha\beta}| = 0$. This equation has a real root $\sigma = c$ at P. Put $e_{\alpha\beta} = g_{\alpha\beta} - c h_{\alpha\beta}$. Then $e_{\alpha\beta} \neq 0$ since (1) is distinct from the energy integral. We can now replace (1) by the integral

$$\tfrac{1}{2} e_{\alpha\beta} \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} + V - cW = \text{const.},$$

since the condition $e_{\alpha\beta,\gamma} = 0$ is satisfied.

Hence assume det. $|h_{\alpha\beta}| = 0$ at P. The matrix $||h_{\alpha\beta}||$ will have constant rank $r < n$ since $h_{\alpha\beta,\gamma} = 0$; also $r > 0$ since otherwise the quantities $h_{\alpha\beta}$ would vanish identically. Hence $h_{\alpha\beta}\xi^\beta = 0$ defines a parallel vector space V^{n-r} in the Riemann space having the tensor g with components $g_{\alpha\beta}$ as its metric tensor.[3] Let V^r be the parallel vector space normal to V^{n-r}. Now introduce canonical coordinates $y^1, \cdots, y^n$ with respect to V^r. Relative to these coordinates the components $g_{\alpha\beta}$ have the form

$$(2) \qquad \begin{cases} g_{\alpha\beta} = a_{\alpha\beta}(y^1, \cdots, y^r), & (\alpha, \beta, = 1, \cdots, r), \\ g_{\alpha\beta} = b_{\alpha\beta}(y^{r+1}, \cdots, y^n), & (\alpha, \beta = r+1, \cdots, n), \\ g_{\alpha\beta} = 0, \text{ otherwise.} \end{cases}$$

It remains to show that the potential function V reduces to the sum of functions $V_1(y^1, \cdots, y^r)$ and $V_2(y^{r+1}, \cdots, y^n)$ in the canonical coordinate system.

The necessary and sufficient conditions for (1) to be a first integral of the dynamical system D are readily observed to be given by $h_{\alpha\beta,\gamma} + h_{\beta\gamma,\alpha} + h_{\gamma\alpha,\beta} = 0$ and

$$(3) \qquad W_{,\alpha} = h_{\alpha\rho} g^{\rho\sigma} V_{,\sigma}.$$

[3] For a complete discussion of parallel vector spaces and canonical coordinates with respect to such spaces, see T. Y. Thomas, *The decomposition of Riemann spaces in the large*, Monatshefte für Mathematik und Physik, vol. 47, 1939, p. 388.

We need here only the condition (3). Differentiating (3) covariantly with respect to x^β, interchanging the indices α and β and subtracting, we find

$$h_{\alpha\mu}g^{\mu\nu}V_{,\nu\beta} - h_{\beta\mu}g^{\mu\nu}V_{,\nu\alpha} = 0, \tag{4}$$

since $h_{\alpha\beta,\gamma} = 0$. But relative to the y coordinates the vector spaces V^r and V^{n-r} are given by $\xi^\alpha = 0$ $(\alpha = r+1, \cdots, n)$ and $\xi^\alpha = 0$ $(\alpha = 1, \cdots, r)$ respectively. Hence with respect to these coordinates the matrix $\| h_{\alpha\beta} \|$ has the form

$$\left\| \begin{array}{ccc|c} h_{11} & \cdots & h_{1r} & \\ \cdots & \cdots & \cdots & 0 \\ h_{r1} & \cdots & h_{rr} & \\ \hline & 0 & & 0 \end{array} \right\|.$$

Now in (4) suppose that α has the range $1, \cdots, r$ and β the range $r+1, \cdots, n$. Then the second set of terms in (4) vanishes identically in the canonical coordinate system and the equation reduces to $h_{\alpha\mu}g^{\mu\nu}V_{,\nu\beta} = 0$. But this implies $V_{,\alpha\beta} = 0$ where $\alpha = 1, \cdots, r$ and $\beta = r+1, \cdots, n$ since the rth order determinant in the upper left hand corner of the above matrix is different from zero and since the g's have the form (2). Writing the latter equation in full we have

$$V_{,\alpha\beta} = \frac{\partial^2 V}{\partial x^\alpha \partial x^\beta} - V_{,\sigma}\Lambda^\sigma_{\alpha\beta} = 0, \qquad \begin{pmatrix} \alpha = 1, \cdots, r \\ \beta = r+1, \cdots, n \end{pmatrix} \tag{5}$$

where we have used $\Lambda^\sigma_{\alpha\beta}$ to denote the Christoffel symbols in the y system. But the Λ's appearing in (5) vanish identically in consequence of (2); hence

$$V = V_1(y^1, \cdots, y^r) + V_2(y^{r+1}, \cdots, y^n)$$

and the proof is complete.

INDIANA UNIVERSITY

TABLE OF THE BESSEL FUNCTIONS AND DERIVATIVES J_2, J_1', J_2', N_2, N_1', N_2'.*

By H. B. Dwight**

x	$J_2(x)$	$J_1'(x)$	$J_2'(x)$	$N_2(x)$	$N_1'(x)$	$N_2'(x)$
.01	.000 012 500	.499 981 25	.002 500 0	−12732.714	6364.854	2 546 479
.02	.000 049 998	.499 925 00	.004 999 7	−3183.417	1590.427	318 310
.03	.000 112 49	.499 831 26	.007 498 9	−1415.029	706.362	94 314.0
.04	.000 199 97	.499 700 03	.009 997 3	−796.094	396.986	39 788.7
.05	.000 312 43	.499 531 33	.012 495	−509.615	253.818	20 371.8
.06	.000 449 87	.499 325 17	.014 991	−353.997	176.067	11 789.2
.07	.000 612 25	.499 081 56	.017 486	−260.164	129.200	7 424.10
.08	.000 799 57	.498 800 53	.019 979	−199.264	98.792 8	4 973.56
.09	.001 011 82	.498 482 10	.022 470	−157.510	77.954 1	3 493.07
.10	.001 248 96	.498 126 30	.024 958	−127.645	63.055 3	2 546.44
.11	.001 510 98	.497 733 16	.027 445	−105.548	52.037 5	1 913.16
.12	.001 797 84	.497 302 70	.029 928	−88.741 1	43.662 5	1 473.61
.13	.002 109 53	.496 834 97	.032 409	−75.661 8	37.148 8	1 159.02
.14	.002 446 00	.496 330 00	.034 886	−65.283 9	31.984 0	927.965
.15	.002 807 23	.495 787 84	.037 360	−56.911 7	27.820 4	754.459
.16	.003 193 18	.495 208 53	.039 830	−50.059 7	24.415 6	621.643
.17	.003 603 81	.494 592 12	.042 296	−44.381 1	21.596 2	518.257
.18	.004 039 08	.493 938 66	.044 757	−39.622 5	19.235 8	436.580
.19	.004 498 94	.493 248 20	.047 215	−35.595 3	17.240 1	371.200
.2	.004 983 35	.492 520 81	.049 667 3	−32.157 14	15.538 0	318.248
.3	.011 165 86	.483 230 19	.073 879 7	−14.480 09	6.836 41	94.240 9
.4	.019 734 66	.470 331 78	.097 353 3	−8.298 34	3.846 16	39.710 8
.5	.030 604 02	.453 932 89	.119 852 4	−5.441 371	2.498 43	20.294 0
.6	.043 665 10	.434 169 88	.141 150 7	−3.892 795	1.792 14	11.715 6
.7	.058 786 94	.411 206 97	.161 033 0	−2.961 478	1.385 41	7.358 11
.8	.075 817 76	.385 234 80	.179 297 6	−2.358 558	1.135 88	4.918 25
.9	.094 586 30	.356 468 75	.195 757 8	−1.945 910	.975 769	3.451 12
1.0	.114 903 48	.325 147 10	.210 243 6	−1.650 683	.869 470	2.520 15
1.1	.136 564 15	.291 528 93	.222 603 9	−1.431 471	.796 817	1.904 56
1.2	.159 349 02	.255 891 86	.232 707 4	−1.263 311	.745 697	1.484 38
1.3	.183 026 70	.218 529 65	.240 443 7	−1.130 412	.708 474	1.190 58
1.4	.207 355 90	.179 749 61	.245 725 0	−1.022 391	.680 143	.981 411
1.5	.232 087 67	.139 870 00	.248 486 3	−.932 193 8	.657 321	.830 616
1.6	.256 967 75	.099 217 21	.248 686 2	−.854 899 4	.637 663	.721 046
1.7	.281 738 94	.058 122 96	.246 307 7	−.786 999 1	.619 513	.641 155
1.8	.306 143 54	.016 921 44	.241 357 5	−.725 948 2	.601 690	.582 944
1.9	.329 925 73	−.024 053 58	.233 866 8	−.669 878 7	.583 349	.540 730

* $N_n(x)$ is the same as $Y_n(x)$ of Mathematical Tables, Vol. VI, of the British Association for the Advancement of Science.

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TABLE OF THE BESSEL FUNCTIONS AND DERIVATIVES J_2, J_1', J_2', N_2, N_1', N_2' —*Continued*

x	$J_2(x)$	$J_1'(x)$	$J_2'(x)$	$N_2(x)$	$N_1'(x)$	$N_2'(x)$
2.0	.352 834 03	−.064 471 62	.223 890 78	−.617 408 1	.563 892	.510 376
2.1	.374 623 63	−.104 008 32	.211 507 73	−.567 511 5	.542 903	.488 808
2.2	.395 058 69	−.142 348 21	.196 818 79	−.519 431 7	.520 108	.473 698
2.3	.413 914 59	−.179 187 40	.179 946 80	−.472 616 9	.495 346	.463 249
2.4	.430 980 04	−.214 236 18	.161 035 23	−.426 674 0	.468 544	.456 051
2.5	.446 059 06	−.247 221 42	.140 246 86	−.381 335 8	.439 703	.450 987
2.6	.458 972 85	−.277 888 90	.117 762 23	−.336 435 6	.408 883	.447 160
2.7	.469 561 50	−.306 005 44	.093 778 04	−.291 886 9	.376 195	.443 845
2.8	.477 685 50	−.331 360 76	.068 505 32	−.247 669 3	.341 793	.440 452
2.9	.483 227 05	−.353 769 30	.042 167 45	−.203 815 2	.305 863	.436 502
3.0	.486 091 26	−.373 071 61	.014 998 12	−.160 400 4	.268 625	.431 608
3.1	.486 207 01	−.389 135 68	−.012 760 81	−.117 535 5	.230 319	.425 459
3.2	.483 527 70	−.401 857 93	−.040 861 56	−.075 358 7	.191 206	.417 811
3.3	.478 031 69	−.411 163 97	−.069 052 72	−.034 029 6	.151 561	.408 477
3.4	.469 722 57	−.417 009 08	−.097 081 54	.006 276 0	.111 670	.397 324
3.5	.458 629 18	−.419 378 46	−.124 696 29	.045 371 4	.071 825	.384 262
3.6	.444 805 40	−.418 287 19	−.151 648 56	.083 063 2	.032 323	.369 246
3.7	.428 329 66	−.413 779 93	−.177 695 56	.119 155 1	−.006 540	.352 266
3.8	.409 304 31	−.405 930 36	−.202 602 32	.153 451 9	−.044 474	.333 351
3.9	.387 854 71	−.394 840 36	−.226 143 89	.185 762 6	−.081 193	.312 557
4.0	.364 128 15	−.380 638 98	−.248 107 40	.215 903 6	−.116 422	.289 974
4.1	.338 292 48	−.363 481 08	−.268 293 98	.243 701 5	−.149 898	.265 715
4.2	.310 534 70	−.343 545 88	−.286 520 61	.268 995 4	−.181 373	.239 920
4.3	.281 059 23	−.321 035 17	−.302 621 78	.291 639 5	−.210 618	.212 747
4.4	.250 086 10	−.296 171 44	−.316 451 02	.311 504 9	−.237 421	.184 378
4.5	.217 848 98	−.269 195 75	−.327 882 20	.328 481 6	−.261 593	.155 005
4.6	.184 593 11	−.240 365 46	−.336 810 71	.342 479 6	−.282 970	.124 841
4.7	.150 573 03	−.209 951 91	−.343 154 37	.353 430 8	−.301 409	.094 105
4.8	.116 050 39	−.178 237 86	−.346 854 19	.361 289 3	−.316 797	.063 028
4.9	.081 291 52	−.145 514 93	−.347 874 88	.366 032 8	−.329 044	.031 846
5.0	.046 565 12	−.112 080 94	−.346 205 18	.367 663	−.338 090	.000 798
5.1	.012 139 77	−.078 237 26	−.341 857 89	.366 205	−.343 904	−.029 873
5.2	−.021 718 41	−.044 286 02	−.334 869 77	.361 709	−.346 480	−.059 928
5.3	−.054 748 15	−.010 527 48	−.325 301 16	.354 248	−.345 843	−.089 131
5.4	−.086 695 38	.022 742 64	−.313 235 39	.343 919	−.342 043	−.117 250
5.5	−.117 315 48	.055 235 81	−.298 778 04	.330 841	−.335 161	−.144 064
5.6	−.146 375 47	.086 673 18	−.282 055 88	.315 156	−.325 300	−.169 361
5.7	−.173 656 04	.116 788 02	−.263 215 74	.297 026	−.312 592	−.192 943
5.8	−.198 953 51	.145 328 04	−.242 423 08	.276 631	−.297 189	−.214 624
5.9	−.222 081 64	.172 057 50	−.219 860 53	.254 170	−.279 268	−.234 237

TABLE OF THE BESSEL FUNCTIONS AND DERIVATIVES J_2, J_1', J_2', N_2, N_1', N_2' —*Continued*

x	$J_2(x)$	$J_1'(x)$	$J_2'(x)$	$N_2(x)$	$N_1'(x)$	$N_2'(x)$
6.0	−.242 873 21	.196 759 23	−.195 726 12	.229 858	−.259 026	−.251 630
6.1	−.261 181 51	.219 236 47	−.170 231 49	.203 923	−.236 679	−.266 672
6.2	−.276 881 60	.239 314 41	−.143 599 92	.176 606	−.212 458	−.279 253
6.3	−.289 871 35	.256 841 68	−.116 064 29	.148 157	−.186 609	−.289 284
6.4	−.300 072 33	.271 691 47	−.087 864 91	.118 836	−.159 392	−.296 696
6.5	−.307 430 39	.283 762 50	−.059 247 34	.088 907	−.131 075	−.301 447
6.6	−.311 916 14	.292 979 75	−.030 460 12	.058 636	−.101 931	−.303 516
6.7	−.313 525 07	.299 294 90	−.001 752 54	.028 293	−.072 242	−.302 905
6.8	−.312 277 56	.302 686 58	.026 627 68	−.001 856	−.042 289	−.299 641
6.9	−.308 218 58	.303 160 31	.054 436 62	−.031 549	−.012 353	−.293 773
7.0	−.301 417 22	.300 748 25	.081 436 38	−.060 527	.017 288	−.285 374
7.1	−.291 965 95	.295 508 67	.107 397 20	−.088 542	.046 362	−.257 665
7.2	−.279 979 74	.287 525 22	.132 099 57	−.115 357	.074 604	−.261 379
7.3	−.265 594 91	.276 905 93	.155 336 16	−.140 745	.101 759	−.246 034
7.4	−.248 967 83	.263 782 03	.176 913 70	−.164 496	.127 588	−.228 657
7.5	−.230 273 41	.248 306 53	.196 654 67	−.186 414	.151 864	−.209 418
7.6	−.209 703 47	.230 652 65	.214 398 89	−.206 324	.174 376	−.188 505
7.7	−.187 464 93	.211 012 03	.230 004 90	−.224 066	.194 934	−.166 119
7.8	−.163 777 84	.189 592 82	.243 351 19	−.239 505	.213 366	−.142 473
7.9	−.138 873 39	.166 617 62	.254 337 22	−.252 526	.229 524	−.117 790
8.0	−.112 991 72	.142 321 26	.262 884 28	−.263 037	.243 279	−.092 301
8.1	−.086 379 73	.116 948 59	.268 936 10	−.270 968	.254 529	−.066 243
8.2	−.059 288 81	.090 752 06	.272 459 28	−.276 274	.263 196	−.039 857
8.3	−.031 972 53	.063 989 32	.273 443 53	−.278 936	.269 226	−.013 384
8.4	−.004 684 34	.036 920 80	.271 901 59	−.278 956	.272 589	.012 934
8.5	.022 324 74	.009 807 26	.267 869 08	−.276 362	.273 284	.038 858
8.6	.048 808 37	−.017 092 69	.261 404 06	−.271 206	.271 332	.064 155
8.7	.074 527 11	−.043 524 92	.252 586 36	−.263 560	.266 780	.088 599
8.8	.099 250 55	−.069 242 18	.241 516 76	−.253 521	.259 698	.111 974
8.9	.122 759 40	−.094 006 32	.228 315 99	−.241 208	.250 182	.134 073
9.0	.144 847 34	−.117 590 48	.213 123 49	−.226 756	.238 346	.154 705
9.1	.165 322 91	−.139 781 07	.196 096 04	−.210 322	.224 329	.173 690
9.2	.184 011 12	−.160 379 75	.177 406 24	−.192 078	.208 286	.190 869
9.3	.200 754 96	−.179 205 07	.157 240 82	−.172 213	.190 391	.206 096
9.4	.215 416 72	−.196 094 15	.135 798 86	−.150 928	.170 836	.219 248
9.5	.227 879 15	−.210 903 95	.113 289 87	−.128 436	.149 823	.230 219
9.6	.238 046 39	−.223 512 55	.089 931 81	−.104 960	.127 570	.238 926
9.7	.245 844 69	−.233 820 08	.065 949 02	−.080 728	.104 302	.245 305
9.8	.251 222 98	−.241 749 51	.041 570 09	−.055 977	.080 252	.249 317
9.9	.254 153 19	−.247 247 15	.017 025 75	−.030 944	.055 661	.250 944
10.0	.254 630 31	−.250 283 04	−.007 453 32	−.005 868	.030 770	.250 189

A THREE-DIMENSIONAL ANALOG OF A PLANE KEMPE LINKAGE

By Michael Goldberg*

1. Introduction. The following remarkable theorem is due to A. B. Kempe:—Given any plane quadrilateral linkage EFGH, one may, without restricting the deformability of the quadrilateral, join a point S by links to properly selected points L, M, N, O on the sides. Extensions and further studies on this theorem were made by Kempe [1] and Darboux [2]. These linkages are illustrated in Figs. 1 and 2.

This paper is concerned with an analogous theorem in three dimensions. We propose to investigate under what conditions it is possible for a skew quadrilateral EFGH composed of four rigid bodies, each of which is hinged to its two adjacent members, to be deformable and to have the further property that four links with a common hinge at S have their other hinges on the sides of the quadrilateral without restricting the deformability of the quadrilateral.

The linkage mechanisms presented here are a natural extension of those previously described by the author [3]. However, they are not the same as the three-dimensional generalization of Kempe linkages described by Fontené [4]. The linkages of Fontené employ only ball-and-socket joints and have two degrees of freedom, while the linkages here described employ only hinged joints and have only one degree of freedom.

2. Four-bar linkage mechanisms in three dimensions. It was first shown by Delassus [5], and more recently in a neater demonstration by Dimentberg and Shor [8], that the only movable hinged four-bar mechanisms in three dimensions are the following:—

(a) the prism linkage in which all the hinges are parallel. This corresponds to the plane linkage.

(b) the pyramidal or spherical linkage in which all the hinges pass through a point. This is the obvious generalization of the plane linkage; a spherical quadrilateral is used instead of a plane quadrilateral.

(c) the skew isogram mechanism due to G. T. Bennett. In this mechanism the opposite links are equal in length, the angle (or twist) between the non-intersecting hinges in a link are equal to the corresponding angle of the opposite link, and the lengths of the links are proportional to the sines of their twists.

3. The Bennett linkage. Consider a four-bar linkage EFGH in which the opposite links are equal in length and all the joints are ball-and-socket joints. This linkage can be made to assume the form of a skew quadrilateral in space. Two typical cases are shown in Figs. 3a and 3b. Let the length of one pair of opposite links be m, and the length of the other opposite pair of links be n.

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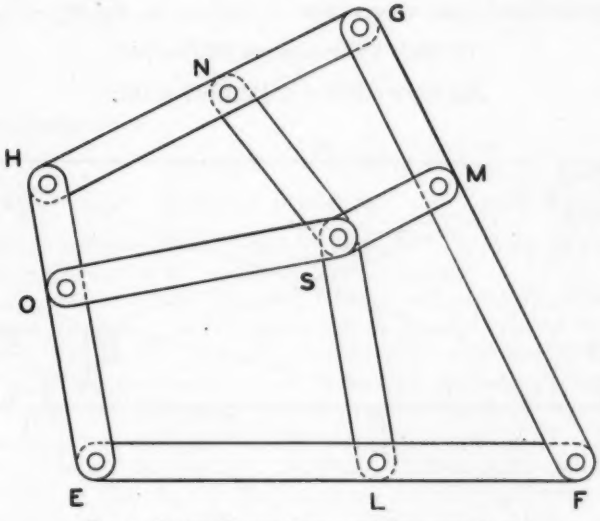

FIG. 1. PLANE KEMPE LINKAGE (STRAIGHT CASE)

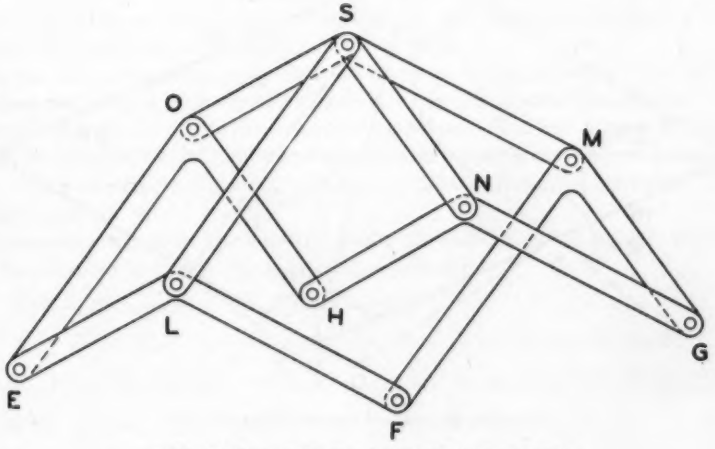

FIG. 2. PLANE KEMPE LINKAGE (BENT CASE)

If the distances EG and FH are allowed to vary so that the four-bar linkage remains symmetrical about a line, then

$$\text{angle FEH} = \text{angle FGH} \equiv 2P$$

$$\text{and angle EHG} = \text{angle EFG} \equiv 2R.$$

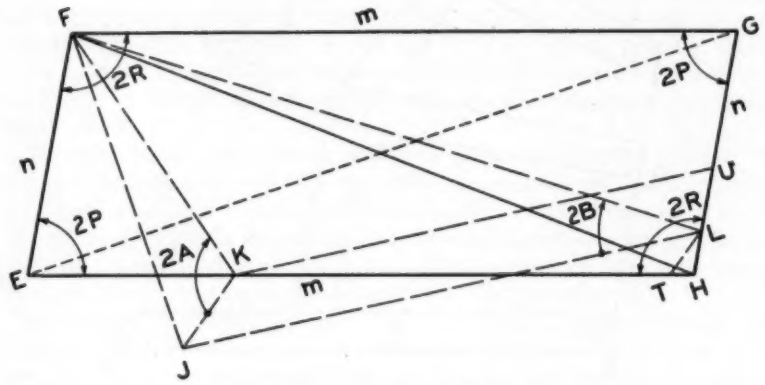

FIG. 3a. BENNETT LINKAGE DERIVATION

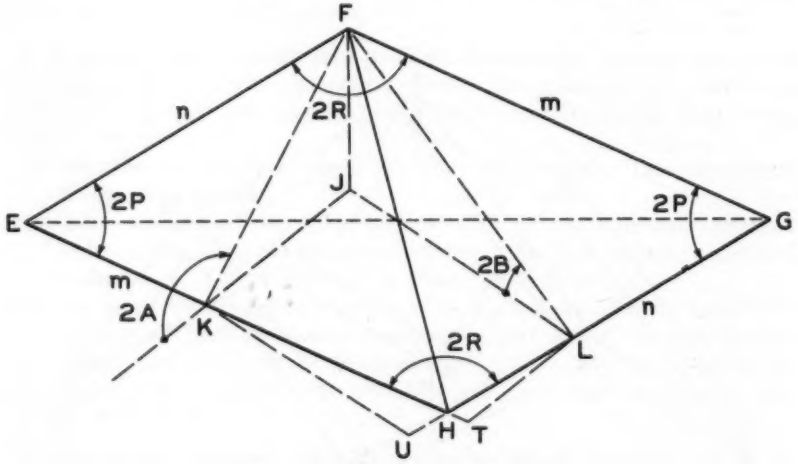

FIG. 3b. BENNETT LINKAGE DERIVATION

Drop the perpendicular from F upon the edge EH whose length is m. Designate by $2A$ the angle that this line makes with the plane EGH. Drop the perpendicular from F upon the line HG whose length is n. Designate by $2B$ the

angle that this line makes with the plane EGH. Then the volume V of the tetrahedron EFGH can be expressed in two ways:

$$6V = (mn \sin 2R)\, n \sin 2P \sin 2A$$

$$\text{or } 6V = (mn \sin 2P)\, m \sin 2R \sin 2B,$$

from which it follows that

$$n \sin 2A = m \sin 2B \text{ or}$$

$$m/\sin 2A = n/\sin 2B. \tag{1}$$

This relation is independent of angles $2P$ and $2R$. Therefore, the angles $2A$ and $2B$ can be fixed while $2P$ and $2R$ are allowed to vary.

Let us now replace the ball-and-socket joint at each vertex by a hinged joint connecting the two links. Let the hinge axis be perpendicular to the plane of the two links. The hinge axis at E is perpendicular to the plane HEF, and the hinge axis at H is perpendicular to the plane EHG. The angle between the planes HEF and EHG is equal to $2A$. Therefore, in the link EH, the angle between the hinge axis at E and the hinge axis at H is equal to $2A$. This angle $2A$ is called the *twist* of the link EH. Similarly, the twist of the link FG is also $2A$. Note that in Figs. 3a and 3b, twists are taken as increasing clockwise from a nearby hinge to a remote hinge when looking along a link. In Fig. 3a both twists are acute. In Fig. 3b, twist $2A$ is obtuse while $2B$ is acute.

From equation (1) it follows that if the lengths m and n are constant, and the twist $2A$ is constant, then the twist $2B$ is constant. This shows that it is possible to construct a movable hinged four-bar linkage in which the hinges are neither parallel nor concurrent. This linkage was described in 1903 by G. T. Bennett [6], a British mathematician.

In the Bennett linkage, note that the *length* of a link is the shortest distance between the hinge axes in that link since it is their common perpendicular. The Bennett linkage may then be described as a space four-bar hinged linkage in which the opposite links have equal lengths and equal twists subject to equation (1). The two length lines which terminate on a hinge line must intersect. They cannot have an offset.

The relation between the variable angles $2P$ and $2R$ as the linkage is moved can be obtained from Fig. 3a or Fig. 3b as follows:

$$\mathrm{KH} = \mathrm{HL} \cos 2R + \mathrm{FL} \cos 2B \sin 2R,$$

$$m - n \cos 2P = (n - m \cos 2P)\cos 2R + m \sin 2P \cos 2B \sin 2R,$$

$$m(1 + \cos 2P \cos 2R) - n(\cos 2P + \cos 2R) = m \cos 2B \sin 2P \sin 2R. \tag{2}$$

Similarly

$$\mathrm{HL} = \mathrm{KH} \cos 2R - \mathrm{FK} \cos 2A \sin 2R,$$

$$n - m \cos 2P = (m - n \cos 2P)\cos 2R - n \sin 2P \cos 2A \sin 2R,$$

$$n(1 + \cos 2P \cos 2R) - m(\cos 2P + \cos 2R) = -n \cos 2A \sin 2P \sin 2R. \tag{3}$$

Adding (2) and (3)

$$(m + n)(1 - \cos 2P)(1 - \cos 2R) = (m \cos 2B - n \cos 2A) \sin 2P \sin 2R,$$

$$\left(\frac{1 - \cos 2P}{\sin 2P}\right)\left(\frac{1 - \cos 2R}{\sin 2R}\right) = \frac{m\cos 2B - n \cos 2A}{m + n}.$$

Using equation (1) for reducing the right hand member, this becomes

$$\tan P \tan R = \frac{\sin (2A - 2B)}{\sin 2A + \sin 2B} \equiv \frac{\sin (A-B)}{\sin (A+B)}$$

$$\equiv \frac{\tan A - \tan B}{\tan A + \tan B} = \frac{a - b}{a + b} \equiv k (a \text{ constant}), \tag{4.1}$$

where $a \equiv \tan A$ and $b \equiv \tan B$.

The internal angles $2P$ and $2R$ are simultaneously each less than 180 degrees or simultaneously each greater than 180 degrees. Therefore, the product tan P

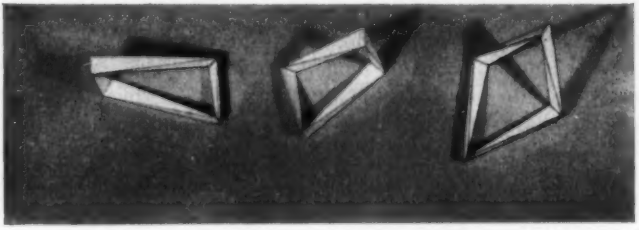

FIG. 4a. MODELS OF BENNETT LINKAGES

tan R is essentially positive. Furthermore, it is convenient to take all twists in the same sense. Since a twist of 180 degrees is equivalent to parallelism, twists cannot exceed 180 degrees. However, the angle $(2A - 2B)$ may be negative and make equation (4.1) inconsistent. This contradiction may be avoided by using absolute values in rewriting equation (4.1) as follows:—

$$\tan P \tan R = \left|\frac{a - b}{a + b}\right| \equiv k \qquad (a \equiv \tan A, \quad b \equiv \tan B). \tag{4.2}$$

Several formulas (apparently different, yet correct) have been published for the value of k. These differences arise from the conventions that have been used for measuring twists and other angles between lines, whether internal or external angles are taken, and whether angles are taken as positive or negative. In this paper, all twists are considered positive and are measured in the same sense, and ranging from zero degrees to all values less than 180 degrees.

Fig. 4a shows several paper models of Bennett linkages. Each link is a

tetrahedron. A pair of opposite edges of the tetrahedron serve as hinges joining the link to the adjacent links.

Fig. 4b is a more practical form of a Bennett linkage. It illustrates how the continuous rotation of a shaft can be transmitted to another non-parallel shaft by means of a hinged connecting rod.

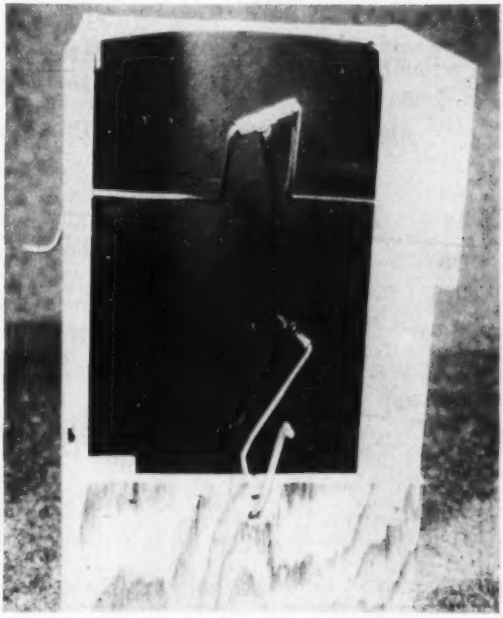

FIG. 4b. APPLICATION OF BENNETT LINKAGE

4. The four-cell division of the Bennett linkage (Conoidal case). Consider a Bennett linkage EFGH. Let four links SL, SM, SN and SO having a common hinge at S be joined to the sides of the Bennett linkage by hinges at the points L, M, N and O which lie on the sides. Now let each of the four cells into which the Bennett linkage is divided be itself a Bennett linkage. (See Fig. 5.) Then the links EL, OS and HN have equal twist. Let this twist be designated by the angle $2A$. Similarly, the links LF, SM, NG have twist $2B$; links EO, LS, FM have twist $2C$; and links OH, SN, MG have twist $2D$.

The opposite angles of a Bennett linkage are equal. Therefore, if the angles at E and G are each equal to $2P$, two of the angles at S are also each equal to $2P$. Similarly, the angles at H, F and the other two angles at S are each equal to $2R$. If one of the angles at L is $2Q$, the other angle is $\pi - 2Q$. Therefore, the opposite angle at M is also $\pi - 2Q$ and the other angle at M is $2Q$. The same pair of angles is found at N and O.

In the Bennett linkage ELSO, from equation (4.2),

$$k_1 \equiv \tan P \tan Q = \left|\frac{\sin (A - C)}{\sin (A + C)}\right| \equiv \left|\frac{a - c}{a + c}\right|, \tag{5}$$

where $a \equiv \tan A$ and $c \equiv \tan C$.

Similarly, in the Bennett linkage SMGN,

$$k_2 \equiv \tan P \tan Q = \left|\frac{\sin (B - D)}{\sin (B + D)}\right| \equiv \left|\frac{b - d}{b + d}\right|, \tag{6}$$

where $b \equiv \tan B$ and $d \equiv \tan D$.

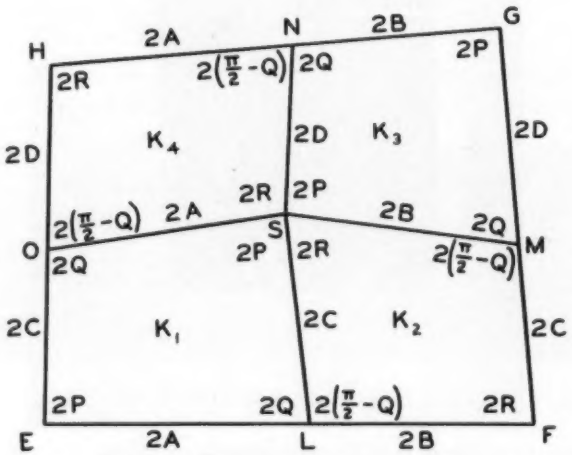

FIG. 5. CONOIDAL FOUR-CELL DIVISION (SCHEMATIC)

By equating (5) and (6) one obtains two solutions (7.1) and (7.2):

$$\frac{a - c}{a + c} = \frac{b - d}{b + d} \quad \text{or} \quad ad = bc, \tag{7.1}$$

$$\frac{a - c}{a + c} = -\frac{b - d}{b + d} \quad \text{or} \quad ab = cd. \tag{7.2}$$

Similarly, using the Bennett linkages LFMS and OSNH, two solutions (8.1) and (8.2) arise:

$$bd = ac, \tag{8.1}$$

$$ab = cd. \tag{8.2}$$

From the Bennett linkage EFGH, using equation (4.2)

$$k \equiv \tan P \tan R = \left|\frac{\tan A' - \tan B'}{\tan A' + \tan B'}\right|,$$

$$\text{where } \tan A' \equiv \tan (A + B) \equiv \frac{a + b}{1 - ab}$$

$$\text{and } \tan B' \equiv \tan (C + D) \equiv \frac{c + d}{1 - cd}.$$

But $k \equiv \tan P \tan R = k_1 k_2$. Substituting for k, k_1 and k_2 in terms of a, b, c and d we obtain

$$\left|\frac{\left(\frac{a+b}{1-ab}\right) - \left(\frac{c+d}{1-cd}\right)}{\left(\frac{a+b}{1-ab}\right) + \left(\frac{c+d}{1-cd}\right)}\right| = \left|\left(\frac{a-c}{a+c}\right)\left(\frac{b-c}{b+c}\right)\right|,$$

which reduces to

$$(a + c)(b + c)(a + b)(1 - cd) - (a + c)(b + c)(1 - ab)(c + d)$$
$$= \pm [(a - c)(b - c)(a + b)(1 - cd) + (a - c)(b - c)(1 - ab)(c + d)].$$

Using the minus sign we obtain

$$2c(a + b)^2(1 - cd) - 2(ab + c^2)(c + d)(1 - ab) = 0. \tag{9.1}$$

Using the plus sign we obtain

$$2(ab + c^2)(a + b)(1 - cd) - 2c(a + b)(c + d)(1 - ab) = 0,$$

which reduces to $(ab - cd)(1 + c^2)(a + b) = 0$. Further, this yields

$$ab = cd, \qquad \text{since } (1 + c^2) \neq 0 \tag{9.2}$$

and the factor $(a + b) = 0$ would make the linkage plane.

Using equation (1) on the linkage EFGH, and the plus and minus signs since the angle sums may exceed 180 degrees,

$$\frac{\sin 2A + \sin 2B}{\sin (2A + 2B)} = \pm \frac{\sin 2C + \sin 2D}{\sin (2C + 2D)},$$

from which the following equations are obtained:

$$\frac{1 + ab}{1 - ab} = -\frac{1 + cd}{1 - cd} \quad \text{or} \quad abcd = 1, \tag{10.1}$$

$$\frac{1 + ab}{1 - ab} = \frac{1 + cd}{1 - cd} \quad \text{or} \quad ab = cd. \tag{10.2}$$

Equations (7.2), (8.2), (9.2) and (10.2) all agree on

$$ab = cd \quad \text{or} \quad \tan A \tan B = \tan C \tan D. \tag{11}$$

If equation (11) is used in combination with one, two or three of the equations (7.1), (8.1), (9.1) and (10.1), special cases of (11) are obtained. Therefore, the only other new combination is (7.1), (8.1), (9.1) and (10.1) and this leads to

$$a = -b, \qquad c = -d. \tag{12}$$

Equation (12) is ruled out since, according to Section 3, only positive values are admissible. Therefore, the following theorem has been established.

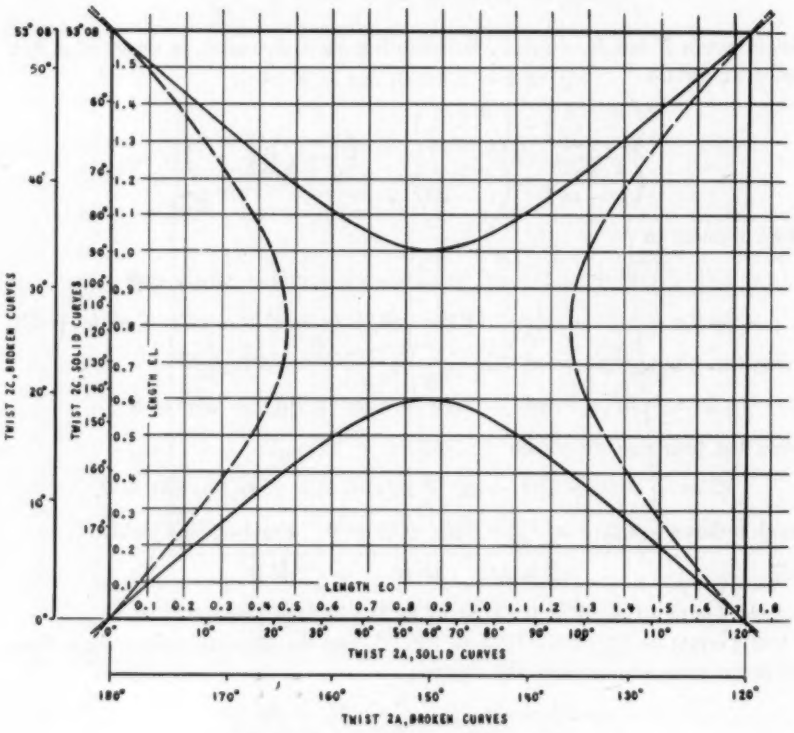

FIG. 6. GRAPH OF FOUR-CELL DIVISIONS

THEOREM. The four links of a movable hinged skew four-bar linkage (a Bennett linkage), of twists $(2A + 2B)$ and $(2C + 2D)$, may be joined by four additional links having a common hinge without restricting the movability of the linkage provided the other ends of the new links divide the original links into segments whose twists are $2A$, $2B$, $2C$ and $2D$ subject to the condition

$$\tan A \tan B = \tan C \tan D,$$

and the lengths of the segments are proportional to $\sin 2A$, $\sin 2B$, $\sin 2C$ and $\sin 2D$.

Since the three hinge axes in an outer link of this four-cell linkage have a common perpendicular, they lie on a conoid. (A conoid is defined as a surface generated by a moving straight line which intersects a fixed straight line at a constant angle.) For this reason this case is called the *conoidal* case.

5. Examples of four-cell divisions. Consider a Bennett linkage which is to be divided into four cells as described in the previous section. As we vary the position of the division point which separates the segments of a link, the segments into which an adjacent link is divided must satisfy equation (11). There may be as many as four real division points on the adjacent links corresponding to the division point on the first link. The totality of four-cell divisions of a

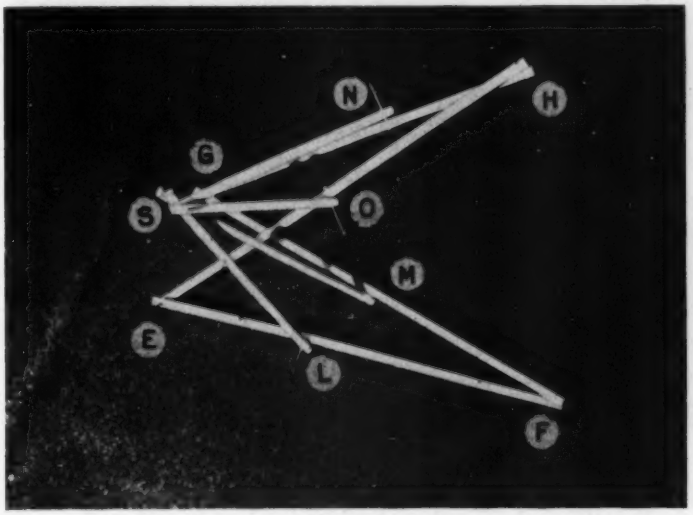

FIG. 7. MODEL OF FOUR-CELL DIVISION—BISECTION OF LONG SIDE

particular Bennett linkage is shown in the graph of Fig. 6. The coordinates of a point on the curve are the lengths of the segments adjacent to a corner hinge. The corresponding twists are given on separate scales on the graph. Note that one angle scale is given for the solid portion of the curves while another angle scale must be used for the broken portions of the curve.

Fig. 7 shows a model of the four-cell division corresponding to the bisection of a long side of the Bennett linkage and it is represented by the middle of the solid curve. Fig. 8 shows a model in which the short side is bisected and it is represented by the middle of the broken curve. Fig. 9 shows several incomplete four-cell divisions, some from the solid portion and some from the broken portion of the curve.

6. The four-cell division of the Bennett linkage (Anconoidal case). Let us consider the more general case of the four-cell division of a Bennett linkage formed by combining four Bennett linkages so that, at the same time, the outer boundary forms another Bennett linkage. In this case, the links EL and LF of Fig. 5 are not aligned. Instead, they make a constant angle as shown in Fig. 10. A similar condition holds for the other sides of the boundary Bennett linkage. This case is called the *anconoidal* case since the outer links may be considered bent like an elbow (an ancon).

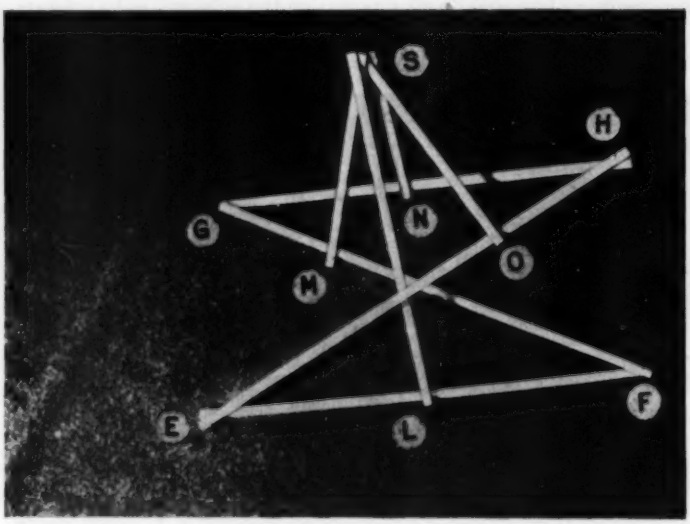

FIG. 8. MODEL OF FOUR-CELL DIVISION—BISECTION OF SHORT SIDE

Let the links EL and LF make a fixed angle $2\psi_1$ at L. Then the equivalent length of the rigid elbow link ELF is the length e of the common perpendicular between the hinges through E and F. Let this perpendicular meet the hinges at the distance x from E, and the distance y from F. Since the lengths EL and LF are proportional to the sines of the twists $2A$ and $2B$, let us take these lengths as $t \sin 2A$ and $t \sin 2B$ where t is the constant of proportionality. Then the length e is given by

$$(13) \quad e^2 = (t \sin 2A \sin \psi_1 + t \sin 2B \sin \psi_1 - x \cos \psi_1 - y \cos \psi_1)^2$$
$$+ (t \sin 2A \cos \psi_1 - t \sin 2B \cos \psi_1 + x \sin \psi_1 - y \sin \psi_1)^2$$
$$+ (x \sin 2A \pm y \sin 2B)^2,$$

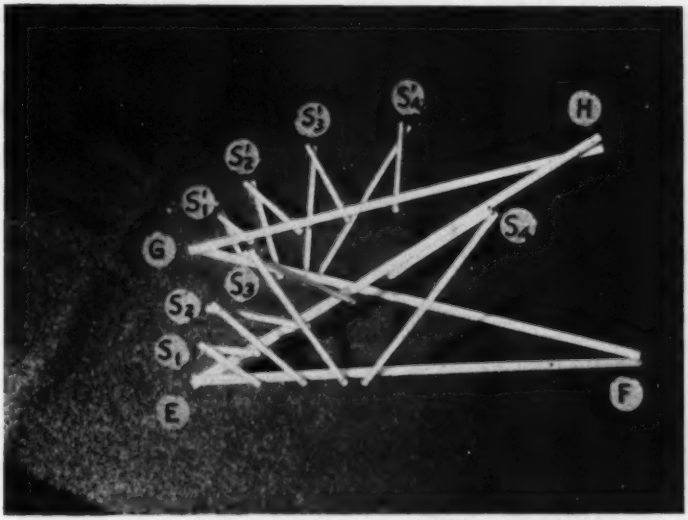

FIG. 9. MODEL OF NEST OF FOUR-CELL DIVISIONS

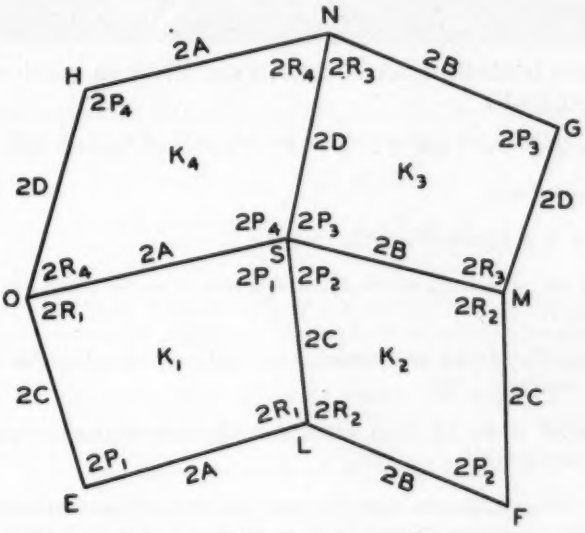

FIG. 10. ANCONOIDAL FOUR-CELL DIVISION (SCHEMATIC)

where the minus sign in the last term is used if the twists $2A$ and $2B$ are both less than 90° or both greater. Since e is the shortest distance between the hinges through E and F, then

$$d(e^2)/dx = 0, \quad \text{and} \quad d(e^2)/dy = 0. \tag{14}$$

Applying equations (14) to (13) gives

$$(1 + \sin^2 2A)x + (\cos 2\psi_1 \pm \sin 2A \sin 2B)y = t \sin 2B \sin 2\psi_1 \tag{15.1}$$

$$(\cos 2\psi_1 \pm \sin 2A \sin 2B)x + (1 + \sin^2 2B)y = t \sin 2A \sin 2\psi_1. \tag{15.2}$$

Solving for x and y

$$x = \frac{t \sin 2\psi_1 [\sin 2A(\cos 2\psi_1 \pm \sin 2A \sin 2B) - \sin 2B(1 + \sin^2 2B)]}{(\cos 2\psi_1 \pm \sin 2A \sin 2B)^2 - (1 + \sin^2 2A)(1 + \sin^2 2B)}, \tag{16.1}$$

$$y = \frac{t \sin 2\psi_1 [\sin 2B(\cos 2\psi_1 \pm \sin 2A \sin 2B) - \sin 2A(1 + \sin^2 2A)]}{(\cos 2\psi_1 \pm \sin 2A \sin 2B)^2 - (1 + \sin^2 2A)(1 + \sin^2 2B)}. \tag{16.2}$$

The elbow link HNG must have the same length and twist as the link ELF. The offset of the common perpendicular at H must also be x, and at G it must be y. However, since the links HOE and GMF must have equal offsets, it follows that x equals y. Therefore, using the plus signs

$$t \sin 2\psi_1 (\sin 2B - \sin 2A)[\cos 2\psi_1 + 1 + (\sin 2A + \sin 2B)^2] = 0. \tag{17.1}$$

Using the minus signs, we obtain

$$t \sin 2\psi_1 (\sin 2B - \sin 2A)[\cos 2\psi_1 + 1 + \sin^2 2A + \sin^2 2B] = 0. \tag{17.2}$$

The factors inside the brackets of (17.1) and (17.2) can vanish only for the conditions of (18.1).

(18.1) $\cos 2\psi_1 = -1$ and $A = B = 0$. This is a degenerate case.

Other solutions are:

(18.2) $t = 0$, a pyramidal case;

(18.3) $\sin 2\psi_1 = 0$, from which $2\psi_1$ equals zero or π, the conoidal case of Section 5;

(18.4) $\sin 2A = \sin 2B$, from which $2A + 2B = \pi$ when the plus sign is used in equation (13);

(18.5) $\sin 2A = \sin 2B$, from which $A = B$ when the minus sign is used in equation (13).

Solution (18.2) makes the outer Bennett linkage a pyramidal linkage although the four cells composing it need not be pyramidal. A model of this anconoidal case of a four-cell linkage is shown in Fig. 11. Note that the hinge axes of the outer Bennett linkage intersect in a point.

Equations similar to (18.4) and (18.5) exist for C and D. These are:

(19.4) $\sin 2C = \sin 2D$ from which $2C + 2D = \pi$ when the plus sign is used in an equation similar to (13);

(19.5) $\sin 2C = \sin 2D$ from which $C = D$ when the minus sign is used in an equation similar to (13).

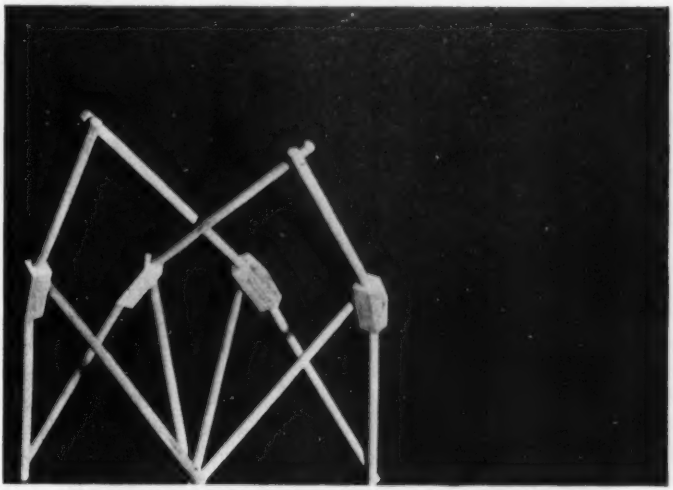

FIG. 11. MODEL OF ANCONOIDAL FOUR-CELL DIVISION

Combining each of the two equations (18.4) and (18.5) with each of the two equations (19.4) and (19.5), three types of solutions are obtained.

(20.1) $\qquad 2A + 2B = \pi, \qquad 2C + 2D = \pi;$

(20.2) $\qquad 2A + 2B = \pi, \qquad C = D;$

(20.3) $\qquad A = B, \qquad C = D.$

The Bennett condition, equation (1), on the outer Bennett linkage imposes further restrictions on the solutions. However, it is satisfied by the following solutions in which $2\psi_2$ is the angle between FM and MG at M.

(21.1) $\qquad 2A = \pi - 2B = 2C = \pi - 2D, \qquad \psi_1 = \psi_2.$

The lengths of the links are equal since

$$\sin 2A = \sin 2B = \sin 2C = \sin 2D.$$

(21.2) $\qquad 2A = 2B = \pi - 2C = \pi - 2D, \qquad \psi_1 = \psi_2.$

The lengths of the links are equal since

$$\sin 2A = \sin 2B = \sin 2C = \sin 2D.$$

The value of k in (4.2) becomes zero; therefore either of R or P must be zero while the other varies.

In solution (21.1), the links of the outer Bennett linkage have equal lengths and equal twists. In this case, it follows from an equation similar to (4.2) that the relative motion of one set of links must be completed before the relative motion of the other set begins.

In solution (21.2), the links of the outer Bennett linkage have equal lengths and supplementary twists. This symmetric condition imposes another condition on the linkage, namely, the opposite angles at S are equal and the sum of the four angles at S is 2π. The linkage then has no freedom of motion. One such rigid network is shown in Fig. 12.

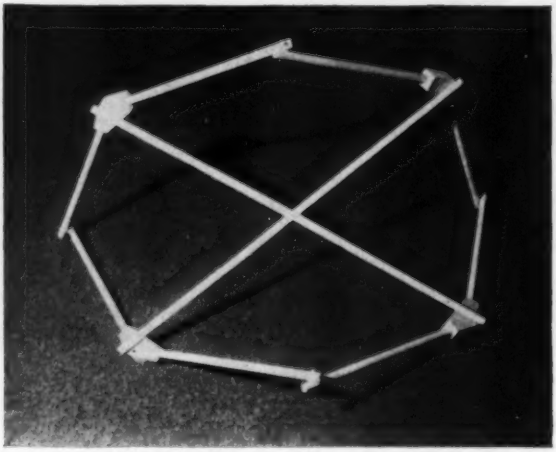

FIG. 12. FOUR-CELL DIVISION OF BENNETT LINKAGE (SYMMETRIC, RIGID)

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ANALYTIC EXPRESSIONS IN PHOTOMETRY AND COLORIMETRY

By Parry Moon* and Domina Eberle Spencer†

1. Introduction

Fundamental in photometry and colorimetry is the equation,

$$L = \int_0^\infty w(\lambda)J(\lambda)d\lambda \tag{1}$$

where $J(\lambda)$ represents the spectral distribution of radiant energy, λ is the wavelength, and $w(\lambda)$ is a weighting function that depends on the properties of the receptor. Equation (1) may be regarded as the definition of a new quantity L, which represents an effect of radiant energy.

The symbol L is used in the generic sense, referring to any effect on any receptor. The effect may be photoelectric, in which case $L = i$, the photoelectric current. The effect may be photochemical or biological—may refer to the photosynthesis of chlorophyl in a leaf, the destruction of bacteria, the production of erythema or the synthesis of vitamin D in human skin.

The most important effect of radiant energy, however, is the visual effect; and thus the customary weighting function in photometry is the *standard lamprosity* (visibility) *function* $v(\lambda)$ for the human eye. This function was adopted by international agreement[1] in 1924. If $J(\lambda)$ is the radiant power per unit area per unit wavelength band (watt m^{-2} micron^{-1}), and $w(\lambda) = v(\lambda)$ is in lumen watt^{-1}, then L represents the visual evaluation of the radiation and is expressed in lumen m^{-2}. This *luminous flux density* or *luminous pharosage*[2] is written

$$D_\ell = \int_0^\infty v(\lambda)J(\lambda)d\lambda. \tag{2}$$

In colorimetry, three standard weighting functions are used instead of the one photometric function, and the resulting three integrals give the *trichromatic specification* of the radiation. In the international trichromatic system, which was adopted[3] in 1931, the three weighting functions are called $\bar{x}(\lambda)$, $\bar{y}(\lambda)$, and $\bar{z}(\lambda)$, and the three integrated quantities are denoted by X, Y, and Z:

$$\begin{aligned} X &= \int_0^\infty \bar{x}(\lambda)J(\lambda)d\lambda, \\ Y &= \int_0^\infty \bar{y}(\lambda)J(\lambda)d\lambda, \\ Z &= \int_0^\infty \bar{z}(\lambda)J(\lambda)d\lambda. \end{aligned} \tag{3}$$

Thus the fundamental quantities of both photometry and colorimetry are obtained from integrals of the form of Eq. (1).

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The various weighting functions have always been defined by tables of numerical values based on experiment. But this method of numerical specification has many obvious disadvantages. It makes extrapolation and interpolation uncertain and allows mathematical manipulation only by clumsy numerical methods. In a series of papers,[4,5,6] we have proposed the use of analytic representations of the standard weighting functions. Such a change makes possible analytic integration and differentiation and eliminates the ambiguity associated with the interpolation and extrapolation of numerically specified functions. An examination of available equations shows[4] that the weighting functions are expressed most conveniently by the Pearson Type V equation,

$$w(\lambda) = A \frac{e^{-q/\lambda}}{\lambda^p}, \tag{4}$$

where A, p, and q are constants that depend on the C.I.E. standardized numerical values.

Frequently the radiation that is to be specified is not received directly from a source but is reflected from a surface having a known spectral reflectance $\rho(\lambda)$. In the past, $\rho(\lambda)$ has been expressed in numerical or graphical form, as has the analogous spectral transmittance $\tau(\lambda)$. We advocate that $\rho(\lambda)$ and $\tau(\lambda)$ be also represented analytically.[6] It is found that the customary reflectance and transmittance curves can be represented by power series, trigonometric series, or by Pearson functions. In this way, a great amount of photometric and colorimetric work, which has always been handled by numerical methods, can be treated analytically.

The purpose of the present paper is to collect and correlate all the necessary equations and numerical values so that the analytic method can be applied advantageously to practical photometry and colorimetry. An integral table is given, and new tables of the various functions are presented. Attention is confined to the discussion of methods that can be used in practice. No detailed comparison of the new data with the present C.I.E. standards will be given, since such comparison has been made elsewhere.[5]

2. Trichromatic Weighting Functions

Analytic expressions were obtained[5] for the standard trichromatic weighting functions $\bar{x}(\lambda)$, $\bar{y}(\lambda)$, and $\bar{z}(\lambda)$. Figure 1 shows graphs of the three analytic representations of the standard data. The $\bar{z}$-function is represented by a single term of the form of Eq. (4). The $\bar{x}$-function requires two terms ($\bar{x}_A$ and $\bar{x}_C$) of this form, to represent the two humps of the curve, and an additional term $\bar{x}_B$ to lower the valley between the peaks. The standard $\bar{y}$-data are represented in a satisfactory manner over most of the wavelength range by one term of the Pearson form, Eq. (4). This is called $\bar{y}_A$. Between 0.63 and 0.67μ, however, a small correction term $\bar{y}_B$ must be introduced; and beyond 0.67μ, another function $\bar{y}_C$ is required.

In all cases, Eq. (4) is employed but the constants are different in each case.

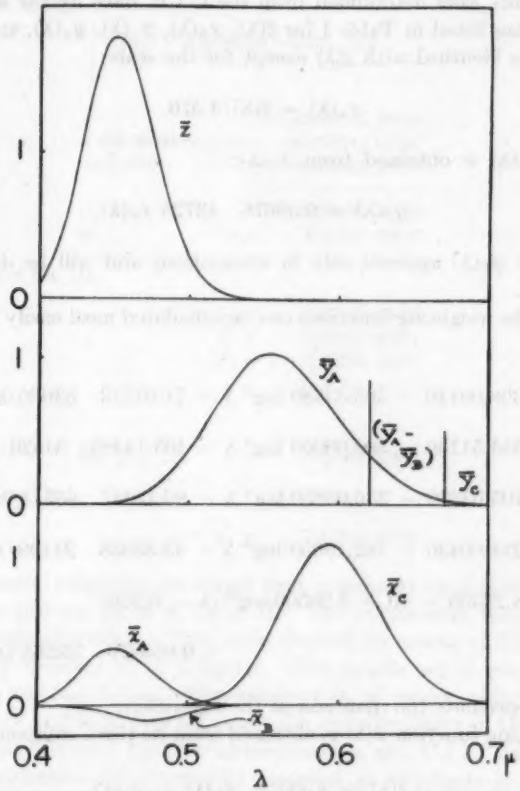

FIG. 1. Calculated curves for $\bar{x}$, $\bar{y}$, and $\bar{z}$. The data for $\bar{z}(\lambda)$ are expressed as a single function having the form of Eq. (4). For $\bar{x}(\lambda)$, three terms of this form are required: $\bar{x}_A$, $\bar{x}_B$, $\bar{x}_C$. The data for $\bar{y}(\lambda)$ are expressed by a single term for wavelengths less than 0.63 μ; but beyond this wavelength other expressions must be employed.

TABLE I

Values of the constants in Eq. (4)

	$\bar{z}(\lambda)$	$\bar{x}_B(\lambda)$	$\bar{x}_C(\lambda)$	$\bar{y}_A(\lambda)$	$\bar{y}_B(\lambda)$	$\bar{y}_D(\lambda)$
p	365.33880	500.00000	336.03850	182.19050	3.20500	1013.50
q	164.95060	237.50000	199.10540	100.93700	0.12820	679.0450
log q	2.21735 38996	2.37566 36140+	2.29908 30387	2.00405 03928	9.10788 80252−10	2.83189 85558
log A	32.75260	54.51150	69.59480	32.41380	5.02860−10	262.03010

$\bar{x}_A(\lambda) = \bar{z}(\lambda)/4.570$, $\bar{y}_C(\lambda) = 0.36678\ 13725\bar{x}_C(\lambda)$

Note: Final zeros indicate that the values are exact by definition to any number of decimals.

These constants were determined from the C.I.E. data by the method of least squares and are listed in Table I for $\bar{z}(\lambda)$, $\bar{x}_B(\lambda)$, $\bar{x}_C(\lambda)$, $\bar{y}_A(\lambda)$, and $\bar{y}_B(\lambda)$. The $\bar{x}_A(\lambda)$ curve is identical with $\bar{z}(\lambda)$ except for the scale:

$$\bar{x}_A(\lambda) = \bar{z}(\lambda)/4.570. \tag{5}$$

Similarly, $\bar{y}_C(\lambda)$ is obtained from $\bar{x}_C(\lambda)$:

$$\bar{y}_C(\lambda) = 0.36678\quad 13725\ \bar{x}_C(\lambda). \tag{6}$$

The quantity $\bar{y}_D(\lambda)$ appears only in integrations and will be discussed in the next section.

Values of the weighting functions can be calculated most easily in the logarithmic form:

$$\begin{aligned}
\log \bar{z}(\lambda) &= 398.09140 - 365.33880 \log^* \lambda - 71.63713\quad 53666/\lambda \\
\log \bar{x}_B(\lambda) &= 554.51150 - 500.00000 \log^* \lambda - 103.14493\quad 94520/\lambda \\
\log \bar{x}_C(\lambda) &= 405.63330 - 336.03850 \log^* \lambda - 86.47037\quad 65371/\lambda \\
\log \bar{y}_A(\lambda) &= 214.60430 - 182.19050 \log^* \lambda - 43.83638\quad 21199/\lambda \\
\log \bar{y}_B(\lambda) &= 8.23360 - 10 - 3.20500 \log^* (\lambda - 0.620) \\
&\qquad - 0.055676\quad 55259/(\lambda - 0.620)
\end{aligned} \tag{7}$$

where log* represents the mantissa of the logarithm.

The weighting function $\bar{x}(\lambda)$ is obtained from its three components,

$$\bar{x}(\lambda) = \bar{x}_A(\lambda) - \bar{x}_B(\lambda) + \bar{x}_C(\lambda). \tag{8}$$

The function $\bar{y}(\lambda)$ is defined as follows:

$$\begin{aligned}
\bar{y}(\lambda) &= \bar{y}_A(\lambda), \quad \lambda \leqq 0.63\mu \\
&= \bar{y}_A(\lambda) - \bar{y}_B(\lambda), \quad 0.63\mu \leqq \lambda \leqq 0.67\mu \\
&= \bar{y}_C(\lambda), \quad 0.67\mu \leqq \lambda.
\end{aligned} \tag{9}$$

Example.

Obtain $\bar{x}$, $\bar{y}$, and $\bar{z}$ for $\lambda = 0.480\mu$.

According to Eq. (7),

$$\log \bar{z}(\lambda) = 398.09140 - 365.33880 \log^* \lambda - 71.63713\quad 53666/\lambda.$$

From a logarithm table,

$$\log^* \lambda = 0.68124\ \ 12373\ \ 556.$$

Thus

$$\begin{aligned} & & +398.09140\ 00000 \\ -365.33880 \log^* \lambda &= & -248.88385\ 61733 \\ -(71.637 \cdots)/\lambda &= & -149.24403\ 20138 \\ \hline \log \bar{z} &= & \bar{1}.96351\ 18129 \\ \text{or} \qquad \bar{z} &= & 0.91941\ 54827 \end{aligned}$$

Similarly, by use of Eq. (7),

$$\begin{aligned} \bar{x}_C &= & 0.00036\ 59306 \\ \bar{x}_A &= & 0.20118\ 50071 \\ -\bar{x}_B &= & -0.10129\ 56482 \end{aligned}$$

and thus by

$$\text{Eq. (8)}, \qquad \bar{x} = 0.10025\ 52895$$

Also, by Eqs. (7) and (9),

$$\bar{y} = \bar{y}_A = 0.14548\ 63529.$$

The trichromatic weighting functions were computed by means of the foregoing equations and are given in Table II. The values were computed to ten figures at intervals of 0.001μ. They were checked by means of fifth differences and were finally rounded off to 8 figures. The results are in general believed to be correct in the last figure. This accuracy is, of course, quite unnecessary for practical work and is far beyond either the reproducibility of the visual data or the agreement between analytic approximation and C.I.E. values. If the analytic representations are eventually accepted as standards in place of the present tabulated results, however, a large number of significant figures will be desirable. Note that the proposed weighting functions, unlike the present C.I.E. functions, are not defined by a set of numerical values. Equation (4), not Table II, constitutes the definition (in conjunction with the numerical constants of Table I).

Table II lists the functions used in colorimetry. The photometric function $v(\lambda)$ is identical with $\bar{y}(\lambda)$ except for the scale. The scale factor is still somewhat in doubt but appears[5] to be 647.8 lumens/young where the lumen is determined from the "new candle"[7] of 1940. There is no necessity to list $v(\lambda)$ separately, since it can be obtained readily from the $\bar{y}$-function of Table II.

A comparison between the new functions and the C.I.E. tabulated values is shown in Figs. 2, 3, and 4. The curve of Fig. 2 gives the analytic representation of $\bar{z}(\lambda)$, while the points represent the C.I.E. values. The agreement is seen to be reasonably satisfactory. Figure 3 presents a corresponding comparison for $\bar{x}(\lambda)$, while Fig. 4 applies to $\bar{y}(\lambda)$.

TABLE II

Calculated values of trichromatic weighting functions

λ (micron)	$\bar{x}(\lambda)$	$\Delta\bar{x}$	$\bar{y}(\lambda)$	$\Delta\bar{y}$	$\bar{z}(\lambda)$	$\Delta\bar{z}$
0.360	^{4}1586 9057		^{3}3044 6763		^{4}7252 1912	
		+462 6096		+948 9020		+2114 1435
1	2049 5153		3993 5783		9366 3347	
		586 3498		1229 5170		2679 6463
2	2635 8651		5223 0953		^{3}1204 5981	
		740 0041		1588 4991		338 1860
3	3375 8692		6811 5944		1542 7841	
		929 9591		2046 4358		424 9975
4	4305 8283		8858 0302		1967 7816	
		1163 7572		2628 9128		531 8463
0.365	^{4}5469 5855		^{7}1148 6943		2499 6279	
		1450 2623		336 7743		662 7837
6	6919 8478		1485 4686		3162 4116	
		1799 8341		430 2183		822 5446
7	8719 6819		1915 6869		3984 9562	
		2224 5191		548 0788		1016 6354
8	^{3}1094 4201		2463 7657		5001 5916	
		273 8252		696 3233		1251 4251
9	1368 2453		3160 0890		6253 0167	
		335 7063		882 2745		1534 2420
0.370	^{3}1703 9516		^{7}4042 3635		7787 2587	
		409 9302		1114 8904		1873 4733
1	2113 8818		5157 2539		9660 7320	
		498 5851		1405 0997		2278 6669
2	2612 4669		6562 3536		^{3}1193 9399	
		604 0355		1766 1969		276 0632
3	3216 5024		8328 5505		1470 0031	
		728 9435		2214 3025		333 1542
4	3945 4459		^{4}1054 2853		1803 1573	
		876 2892		276 8964		400 5023
0.375	^{4}4821 7351		^{4}1331 1817		^{3}2203 6596	
		1049 3909		345 3674		479 6252
6	5871 1260		1676 5491		2683 2848	
		1251 9207		429 6799		572 2626
7	7123 0467		2106 2290		3255 4874	
		1487 9192		533 2328		680 0832
8	8610 9659		2639 4618		3935 5706	
		1761 8061		660 0954		805 2890
9	^{3}1037 2772		3299 5572		4740 8596	
		207 8382		815 1253		950 0184

TABLE II—*Continued*

λ (micron)	$\bar{x}(\lambda)$	$\Delta\bar{x}$	$\bar{y}(\lambda)$	$\Delta\bar{y}$	$\bar{z}(\lambda)$	$\Delta\bar{z}$
0.380	^{3}1245 1154		^{4}4114 6825		^{3}5690 8780	
		244 2835		1004 1058		1116 6462
1	1489 3989		5118 7883		6807 5242	
		286 0723		1233 9047		1307 7182
2	1775 4712		6352 6930		8115 2424	
		333 7975		1512 6556		1525 9525
3	2109 2687		7865 3486		9641 1949	
		388 0851		1849 9681		1774 2192
4	2497 3538		9715 3167		^{1}1141 5414	
		449 5922		2257 1653		205 5534
0.385	^{2}2946 9460		^{5}1197 2482		^{1}1347 0948	
		519 0023		274 7554		237 3037
6	3465 9483		1472 0036		1584 3985	
		597 0203		333 6740		272 9969
7	4062 9686		1805 6776		1857 3954	
		684 3661		404 2966		312 9647
8	4747 3347		2209 9742		2170 3601	
		781 7666		488 7520		357 5421
9	5529 1013		2698 7262		2527 9022	
		889 9478		589 5169		407 0651
0.390	^{2}6419 0491		^{3}3288 2431		^{1}2934 9673	
		1009 6241		709 4663		461 8649
1	7428 6732		3997 7094		3396 8322	
		1141 4881		851 9295		522 2639
2	8570 1613		4849 6389		3919 0961	
		1286 1975		1020 7518		588 5699
3	9856 3588		5870 3907		4507 6660	
		1444 3632		1220 3647		661 0705
4	^{1}1130 0722		7090 7554		5168 7365	
		161 6535		1455 8623		740 0273
0.395	^{1}1291 7257		^{3}8546 6177		^{1}5908 7638	
		180 3184		1733 0853		825 6690
6	1472 0441		^{4}1027 9703		6734 4328	
		200 4694		205 8720		918 1847
7	1672 5135		1233 8423		7652 6175	
		222 1340		244 0390		1017 7175
8	1894 6475		1477 8813		8670 3350	
		245 3269		288 6783		1124 3568
9	2139 9744		1766 5596		9794 6918	
		270 0504		340 7767		1238 1356
0.400	^{1}2410 0248		^{2}2107 3363		^{9}1103 2827	
		296 2895		401 4521		135 9014
1	2706 3143		2508 7884		1239 1841	
		324 0137		471 9689		148 6884
2	3030 3280		2980 7573		1387 8725	
		353 1742		553 7530		162 1561
3	3383 5022		3534 5103		1550 0286	
		383 7024		648 4098		176 2772
4	3767 2046		4182 9201		1726 3058	
		415 5095		757 7408		191 0162

TABLE II—*Continued*

λ (micron)	$\bar{x}(\lambda)$	$\Delta\bar{x}$	$\bar{y}(\lambda)$	$\Delta\bar{y}$	$\bar{z}(\lambda)$	$\Delta\bar{z}$
0.405	^{1}4182 7141		^{1}4940 6609		^{0}1917 3220	
		448 4855		883 7653		206 3278
6	4631 1996		5824 4262		2123 6498	
		482 4983		1028 7392		222 1581
7	5113 6979		6853 1654		2345 8079	
		517 3937		1195 1784		238 4430
8	5631 0916		8048 3438		2584 2509	
		552 9953		1385 8820		255 1094
9	6184 0869		9434 2258		2839 3603	
		589 1041		1603 9572		272 0748
0.410	^{1}6773 1910		^{2}1103 8183		^{0}3111 4391	
		625 5003		185 2844		289 2465
1	7398 6913		1289 1027		3400 6816	
		661 9428		213 6345		306 5253
2	8060 6341		1502 7372		3707 2069	
		698 1715		245 8651		323 8014
3	8758 8056		1748 6023		4031 0083	
		733 9084		282 4373		340 9591
4	9492 7140		2031 0396		4371 9674	
		768 8600		323 8568		357 8755
0.415	^{0}1026 1574		^{2}2354 8964		^{0}4729 8429	
		80 2717		370 6777		374 4217
6	1106 4291		2725 5741		5104 2646	
		83 5165		423 5050		390 4648
7	1189 9456		3149 0791		5494 7294	
		86 5875		482 9981		405 8683
8	1276 5331		3632 0772		5900 5977	
		89 4516		549 8745		420 4935
9	1365 9847		4181 9517		6321 0912	
		92 0756		624 9117		434 2010
0.420	^{0}1458 0603		^{2}4806 8634		^{0}6755 2921	
		94 4268		708 9521		446 8522
1	1552 4871		5515 8155		7202 1443	
		96 4724		802 9051		458 3112
2	1648 9595		6318 7206		7660 4555	
		98 1815		907 7505		468 4457
3	1747 1410		7226 4711		8128 9012	
		99 5239		1024 5410		477 1287
4	1846 6649		8251 0121		8606 0299	
		100 4716		1154 4064		484 2403
0.425	^{0}1947 1365		^{3}9405 4185		^{0}9090 2702	
		100 9988		1298 5545		489 6692
6	2048 1353		^{3}1070 3973		9579 9394	
		101 0822		145 8273		493 3138
7	2149 2175		1216 2246		1.0073 2532	
		100 7013		163 4938		495 0840
8	2249 9188		1379 7184		1.0568 3372	
		99 8393		183 0003		494 9019
9	2349 7581		1562 7187		1.1063 2391	
		98 4828		204 5014		492 7035

TABLE II—*Continued*

λ (micron)	$\bar{x}(\lambda)$	$\Delta\bar{x}$	$\bar{y}(\lambda)$	$\Delta\bar{y}$	$\bar{z}(\lambda)$	$\Delta\bar{z}$
0.430	^{2}2448 2409		^{2}1767 2201		1.1555 9426	
		96 6221		228 1603		488 4395
1	2544 8630		1995 3804		1.2044 3821	
		94 2520		254 1490		482 0758
2	2639 1150		2249 5294		1.2526 4579	
		91 3712		282 6481		473 5942
3	2730 4862		2532 1775		1.3000 0521	
		87 9834		313 8473		462 9936
4	2818 4696		2846 0248		1.3463 0457	
		84 0960		347 9450		450 2893
0.435	^{2}2902 5656		^{2}3193 9698		1.3913 3350	
		79 7218		385 1477		435 5135
6	2982 2874		3579 1175		1.4348 8485	
		74 8777		425 6705		418 7150
7	3057 1651		4004 7880		1.4767 5635	
		69 5850		469 7363		399 9595
8	3126 7501		4474 5243		1.5167 5230	
		63 8695		517 5756		379 3279
9	3190 6196		4992 0999		1.5546 8509	
		57 7612		569 4260		356 9170
0.440	^{2}3248 3808		^{2}5561 5259		1.5903 7679	
		51 2937		625 5314		332 8371
1	3299 6745		6187 0573		1.6236 6050	
		44 5041		686 1418		307 2125
2	3344 1786		6873 1991		1.6543 8175	
		37 4332		751 5124		280 1789
3	3381 6118		7624 7115		1.6823 9964	
		30 1241		821 9020		251 8832
4	3411 7359		8446 6135		1.7075 8796	
		22 6225		897 5742		222 4813
0.445	3434 3584		^{2}9344 1877		1.7298 3609	
		14 9760		978 7933		192 1371
6	3449 3344		^{1}1032 2981		1.7490 4980	
		+7 2336		106 5827		161 0203
7	3456 5680		1138 8808		1.7651 5183	
		−5546		115 8939		129 3058
8	3456 0134		1254 7747		1.7780 8241	
		8 3386		125 8398		97 1705
9	3447 6748		1380 6145		1.7877 9946	
		16 0678		136 4464		64 7934
0.450	^{2}3431 6070		^{1}1517 0609		1.7942 7880	
		−23 6929		147 7397		32 3523
1	3407 9141		1664 8006		1.7975 1402	
		31 1653		159 7450		+0235
2	3376 7488		1824 5456		1.7975 1638	
		38 4385		172 4866		−32 0205
3	3338 3103		1997 0322		1.7943 1433	
		45 4675		185 9887		63 6123
4	3292 8428		2183 0209		1.7879 5310	
		52 2105		200 2734		94 5909

TABLE II—*Continued*

λ (micron)	$\bar{x}(\lambda)$	$\Delta\bar{x}$	$\bar{y}(\lambda)$	$\Delta\bar{y}$	$\bar{z}(\lambda)$	$\Delta\bar{z}$
0.455	^{0}3240 6323		^{1}2383 2943		1.7784 9401	
		−58 6280		215 3624		−124 8030
6	3182 0043		2598 6567		1.7660 1371	
		64 6845		231 2754		154 1040
7	3117 3198		2829 9321		1.7506 0331	
		70 3475		248 0310		182 3596
8	3046 9723		3077 9631		1.7323 6735	
		75 5887		265 6455		209 4460
9	2971 3836		3343 6086		1.7114 2275	
		80 3840		284 1334		235 2509
0.460	^{0}2890 9996		^{1}3627 7420		1.6878 9766	
		−84 7130		303 5071		−259 6740
1	2806 2866		3931 2491		1.6619 3026	
		88 5604		323 7764		282 6279
2	2717 7262		4255 0255		1.6336 6747	
		91 9144		344 9487		304 0379
3	2625 8118		4599 9742		1.6032 6368	
		94 7685		367 0285		323 8422
4	2531 0433		4967 0027		1.5708 7946	
		97 1198		390 0175		341 9922
0.465	^{0}2433 9235		^{1}5357 0202		1.5366 8024	
		−98 9698		413 9139		−358 4519
6	2334 9537		5770 9341		1.5008 3505	
		100 3241		438 7133		373 1987
7	2234 6296		6209 6474		1.4635 1518	
		101 1923		464 4070		386 2213
8	2133 4373		6674 0544		1.4248 9305	
		101 5869		490 9834		397 5212
9	2031 8504		7165 0378		1.3851 4093	
		101 5244		518 4267		407 1107
0.470	1930 3260		^{1}7683 4645		1.3444 2986	
		−101 0241		546 7174		−415 0128
1	1829 3019		8230 1819		1.3029 2858	
		100 1077		575 8320		421 2605
2	1729 1942		8806 0139		1.2608 0253	
		98 7994		605 7431		425 8961
3	1630 3948		9411 7570		1.2182 1292	
		97 1253		636 4190		428 9702
4	1533 2695		^{0}1004 8176		1.1753 1590	
		95 1131		66 7823		430 5407
0.475	1438 1564		^{0}1071 5999		1.1322 6183	
		−92 7916		69 9916		−430 6723
6	1345 3648		1141 5915		1.0891 9460	
		90 1906		73 2653		429 4357
7	1255 1742		1214 8568		1.0462 5103	
		87 3400		76 5986		426 9058
8	1167 8342		1291 4554		1.0035 6045	
		84 2702		79 9861		423 1624
9	1083 5640		1371 4415		^{0}9612 4421	
		81 0111		83 4220		418 2873

TABLE II—*Continued*

λ (micron)	$\bar{x}(\lambda)$	$\Delta\bar{x}$	$\bar{y}(\lambda)$	$\Delta\bar{y}$	$\bar{z}(\lambda)$	$\Delta\bar{z}$
0.480	^{0}1002 5529		^{0}1454 8635		^{0}9194 1548	
		−775 9228		86 9005		−412 3655
1	^{0}9249 6062		1541 7640		8781 7893	
		740 4225		90 4147		405 4833
2	8509 1837		1632 1787		8376 3060	
		703 8855		93 9579		397 7272
3	7805 2982		1726 1366		7978 5788	
		666 5740		97 5227		389 1843
4	7138 7242		1823 6593		7589 3945	
		628 7340		101 1014		379 9407
0.485	^{0}6509 9902		^{0}1924 7607		^{0}7209 4538	
		−590 5948		104 6861		−370 0844
6	5919 3954		2029 4468		6839 3724	
		552 3667		108 2683		359 6895
7	5367 0287		2137 7151		6479 6829	
		514 2399		111 8395		348 8455
8	4852 7888		2249 5546		6130 8374	
		476 3840		115 3907		337 6278
9	4376 4048		2364 9453		5793 2096	
		438 9470		118 9127		326 1109
0.490	^{0}3937 4578		^{0}2483 8580		^{0}5467 0987	
		−402 0550		122 3961		−314 3661
1	3535 4028		2606 2541		5152 7326	
		365 8125		125 8315		302 4612
2	3169 5903		2732 0856		4850 2714	
		330 3016		129 2089		290 4596
3	2839 2887		2861 2945		4559 8118	
		295 5841		132 5186		278 4208
4	2543 7046		2993 8131		4281 3910	
		261 6999		135 7505		266 4002
0.495	^{0}2282 0047		^{0}3129 5636		^{0}4014 9908	
		−228 6699		138 8948		−254 4484
6	2053 3348		3268 4584		3760 5424	
		196 4955		141 9412		242 6123
7	1856 8393		3410 3996		3517 9301	
		165 1600		144 8798		230 9341
8	1691 6793		3555 2794		3286 9960	
		134 6304		147 7007		219 4516
9	1557 0489		3702 9801		3067 5444	
		104 8581		150 3941		208 1988
0.500	^{1}1452 1908		^{0}3853 3742		^{0}2859 3456	
		−75 7800		152 9501		−197 2056
1	1376 4108		4006 3243		2662 1400	
		−47 3209		155 3593		186 4976
2	1329 0899		4161 6836		2475 6424	
		−19 3940		157 6124		176 0969
3	1309 6959		4319 2960		2299 5455	
		+8 0968		159 7003		166 0222
4	1317 7927		4478 9963		2133 5233	
		35 2565		161 6141		156 2884

TABLE II—*Continued*

λ (micron)	$\bar{x}(\lambda)$	$\Delta\bar{x}$	$\bar{y}(\lambda)$	$\Delta\bar{y}$	$\bar{z}(\lambda)$	$\Delta\bar{z}$
0.505	^{1}1353 0492		^{4}4640 6104		^{0}1977 2349	
		62 1961		163 3456		−146 9073
6	1415 2453		4803 9560		1830 3276	
		89 0320		164 8866		137 8881
7	1504 2773		4968 8426		1692 4395	
		115 8849		166 2294		129 2365
8	1620 1622		5135 0720		1563 2030	
		142 8771		167 3671		120 9563
9	1763 0393		5302 4391		1442 2467	
		170 1322		168 2024		113 0488
0.510	^{1}1933 1715		^{5}5470 7315		^{0}1329 1979	
		197 7734		168 9996		−105 5128
1	2130 9449		5639 7311		1223 6851	
		225 9224		169 4825		98 3456
2	2356 8673		5809 2136		1125 3395	
		254 6973		169 7364		91 5426
3	2611 5646		5978 9500		1033 7969	
		284 2130		169 7564		850 9766
4	2895 7776		6148 7064		^{1}9486 9928	
		314 5783		169 5385		790 0345
0.515	^{1}3210 3559		^{6}6318 2449		^{1}8696 9583	
		345 8961		169 0796		−732 5136
6	3556 2520		6487 3245		7964 4447	
		378 2620		168 3768		678 3187
7	3934 5140		6655 7013		7286 1260	
		411 7631		167 4280		627 3473
8	4346 2771		6823 1293		6658 7797	
		446 4774		166 2319		579 4860
9	4792 7545		6989 3612		6079 2937	
		482 4727		164 7877		534 6227
0.520	^{1}5275 2272		^{7}7154 1489		^{1}5544 6710	
		519 8064		163 0955		−492 6372
1	5795 0336		7317 2444		5052 0338	
		558 5240		161 1557		453 4074
2	6353 5576		7478 4001		4598 6264	
		598 6592		158 9699		416 8096
3	6952 2168		7637 3700		4181 8168	
		640 2327		156 5398		382 7195
4	7592 4495		7793 9098		3799 0973	
		683 2524		153 8682		351 0130
0.525	^{1}8275 7019		^{7}7947 7780		^{1}3448 0843	
		727 7121		150 9586		−321 5669
6	9003 4140		8098 7366		3126 5174	
		773 5920		147 8148		294 2594
7	9777 0060		8246 5514		2832 2580	
		820 8580		144 4415		268 9714
8	^{0}1059 7864		8390 9929		2563 2866	
		86 9461		140 8439		245 5862
9	1146 7325		8531 8368		2317 7004	
		91 9339		137 0281		223 9902

TABLE II—*Continued*

λ (micron)	$\bar{x}(\lambda)$	$\Delta\bar{x}$	$\bar{y}(\lambda)$	$\Delta\bar{y}$	(λ)	$\Delta\bar{z}$
0.530	^{0}1238 6664		^{0}8668 8649		^{1}2093 7102	
		97 0412		133 0004		−204 0734
1	1335 7076		8801 8653		1889 6368	
		102 2590		128 7678		185 7299
2	1437 9666		8930 6331		1703 9069	
		107 5765		124 3379		168 8572
3	1545 5431		9054 9710		1535 0497	
		112 9816		119 7188		153 3575
4	1658 5247		9174 6898		1381 6922	
		118 4607		114 9192		139 1370
0.535	^{0}1776 9854		^{0}9289 6090		^{1}1242 5552	
		123 9993		109 9480		−126 1065
6	1900 9847		9399 5570		1116 4487	
		129 5808		104 8145		114 1810
7	2030 5655		9504 3715		1002 2677	
		135 1883		99 5288		1032 8018
8	2165 7538		9603 9003		^{0}8989 8748	
		140 8029		94 1009		933 2773
9	2306 5567		9698 0012		8056 5975	
		146 4051		88 5415		842 5189
0.540	^{0}2452 9618		^{0}9786 5427		^{0}7214 0786	
		151 9743		82 8611		−759 8507
1	2604 9361		9869 4038		6454 2279	
		157 4889		77 0710		684 6375
2	2762 4250		9946 4748		5769 5904	
		162 9266		71 1823		616 2844
3	2925 3516		1.0017 6571		5153 3060	
		168 2644		65 2064		554 2351
4	3093 6160		1.0082 8635		4599 0709	
		173 4787		59 1549		497 9700
0.545	^{0}3267 0947		1.0142 0184		^{0}4101 1009	
		178 5454		53 0394		−447 0052
6	3445 6401		1.0195 0578		3654 0957	
		183 4402		46 8716		400 8911
7	3629 0803		1.0241 9294		3253 2046	
		188 1386		40 6633		359 2001
8	3817 2189		1.0282 5927		2893 9947	
		192 6161		34 4259		321 5752
9	4009 8350		1.0317 0186		2572 4195	
		196 8480		28 1715		287 6288
0.550	4206 6830		1.0345 1901		^{0}2284 7907	
		200 8106		21 9112		−257 0407
1	4407 4936		1.0367 1013		2027 7500	
		204 4798		15 6568		229 5064
2	4611 9734		1.0382 7581		1798 2436	
		207 8325		9 4193		204 7456
3	4819 8059		1.0392 1774		1593 4980	
		210 8463		3 2099		182 5008
4	5030 6522		1.0395 3873		1410 9972	
		213 4995		+2 9806		162 5358

TABLE II—*Continued*

λ (micron)	$\bar{x}(\lambda)$	$\Delta\bar{x}$	$\bar{y}(\lambda)$	$\Delta\bar{y}$	$\bar{z}(\lambda)$	$\Delta\bar{z}$
0.555	^{0}5244 1517		1.0392 4267		^{2}1248 4614	
		215 7718		−9 0813		−144 6342
6	5459 9235		1.0383 3454		1103 8272	
		217 6433		15 1422		1285 9791
7	5677 5668		1.0368 2032		^{3}9752 2929	
		219 0062		21 1331		1142 4605
8	5896 6630		1.0347 0701		8609 8324	
		220 1136		27 0442		1014 1360
9	6116 7766		1.0320 0259		7595 6964	
		220 6803		32 8661		899 5024
0.560	^{0}6337 4569		1.0287 1597		^{4}6696 1940	
		220 7829		−38 5901		−797 1917
1	6558 2398		1.0248 5696		5899 0023	
		220 4094		44 2073		705 9612
2	6778 6492		1.0204 3623		5193 0411	
		219 5498		49 7094		624 6830
3	6998 1990		1.0154 6529		4568 3581	
		218 1964		55 0887		552 3348
4	7216 3954		1.0099 5642		4016 0233	
		216 3428		60 3379		487 9914
0.565	^{0}7432 7382		1.0039 2263		^{3}3528 0319	
		213 9852		−65 4501		−430 8163
6	7646 7234		^{0}9973 7762		3097 2156	
		211 1218		70 4185		380 0539
7	7857 8452		9903 3577		2717 1617	
		207 7528		75 2375		335 0229
8	8065 5980		9828 1202		2382 1388	
		203 8807		79 9013		295 1094
9	8269 4787		9748 2189		2087 0294	
		199 5101		84 4046		259 7607
0.570	^{0}8468 9888		^{0}9663 8143		^{2}1827 2687	
		194 6477		−88 7439		−228 4804
1	8663 6365		9575 0704		1598 7883	
		189 3024		92 9135		200 8224
2	8852 9389		9482 1569		1397 9659	
		183 4854		96 9107		176 3869
3	9036 4243		9385 2462		1221 5790	
		177 2095		100 7322		154 8154
4	9213 6338		9284 5140		1066 7636	
		170 4900		104 3752		1357 8688
0.575	^{0}9384 1238		^{0}9180 1388		^{0}9309 7677	
		163 3437		−107 8375		−1190 1471
6	9547 4675		9072 3013		8119 6206	
		155 7895		111 1173		1042 4262
7	9703 2570		8961 1840		7077 1944	
		147 8479		114 2131		912 4190
8	9851 1049		8846 9709		6164 7754	
		139 5411		117 1243		798 0873
9	9990 6460		8729 8466		5366 6881	
		130 8928		119 8502		697 6159

TABLE II—*Continued*

λ (micron)	$\bar{x}(\lambda)$	$\Delta\bar{x}$	$\bar{y}(\lambda)$	$\Delta\bar{y}$	$\bar{z}(\lambda)$	$\Delta\bar{z}$
0.580	1.0121 5388		^{8}8609 9964		^{4}4669 0722	
		121 9281		−122 3909		−609 3888
1	1.0243 4669		8487 6055		4059 6834	
		112 6734		124 7465		531 9704
2	1.0356 1403		8362 8590		3527 7130	
		103 1560		126 9180		464 0855
3	1.0459 2963		8235 9410		3063 6275	
		93 4044		128 9064		404 6023
4	1.0552 7007		8107 0346		2659 0252	
		83 4480		130 7131		352 5180
0.585	1.0636 1487		^{7}7976 3215		^{2}2306 5072	
		73 3166		−132 3401		−306 9439
6	1.0709 4653		7843 9814		1999 5633	
		63 0407		133 7894		267 0939
7	1.0772 5060		7710 1920		1732 4694	
		52 6511		135 0635		232 2731
8	1.0825 1571		7575 1285		1500 1963	
		42 1789		136 1653		201 8674
9	1.0867 3360		7438 9632		1298 3289	
		31 6552		137 0977		175 3345
0.590	1.0898 9912		^{7}7301 8655		^{1}1122 9944	
		21 1108		−137 8640		−1521 9695
1	1.0920 1020		7164 0015		^{9}9707 9744	
		10 5767		138 4678		1320 3332
2	1.0930 6787		7025 5337		8387 6412	
		+0831		138 9128		1144 7288
3	1.0930 7618		6886 6209		7242 9124	
		−10 3402		139 2032		991 8945
4	1.0920 4216		6747 4177		6251 0179	
		−20 6640		139 3428		858 9627
0.595	1.0899 7576		^{6}6608 0749		^{5}5392 0552	
		−30 8601		−139 3363		−743 4152
6	1.0868 8975		6468 7386		4648 6400	
		40 9008		139 1879		643 0415
7	1.0827 9967		6329 5507		4005 5985	
		50 7596		138 9026		555 9032
8	1.0777 2371		6190 6481		3449 6953	
		60 4121		138 4848		480 3018
9	1.0716 8250		6052 1633		2969 3935	
		69 8334		137 9395		414 7497
0.600	1.0646 9916		^{5}5914 2238		^{2}2554 6438	
		−79 0009		−137 2718		−357 9457
1	1.0567 9907		5776 9520		2196 6981	
		87 8934		136 4863		308 7517
2	1.0480 0973		5640 4657		1887 9464	
		96 4912		135 5885		266 1735
3	1.0383 6061		5504 8772		1621 7729	
		104 7757		134 5834		229 3430
4	1.0278 8304		5370 2938		1392 4299	
		112 7304		133 4760		197 5030

TABLE II—*Continued*

λ (micron)	$\bar{x}(\lambda)$	$\Delta\bar{x}$	$\bar{y}(\lambda)$	$\Delta\bar{y}$	$\bar{z}(\lambda)$	$\Delta\bar{z}$
0.605	1.0166 1000		^{0}5236 8178		^{1}1194 9269	
		−120 3398		−132 2715		−169 9932
6	1.0045 7602		5104 5463		1024 9337	
		127 5906		130 9753		1462 3803
7	^{0}9918 1696		4973 5710		^{0}8786 9572	
		134 4708		129 5921		1257 3689
8	9783 6988		4843 9789		7529 5883	
		140 9702		128 1273		1080 5389
9	9642 7286		4715 8516		6449 0494	
		147 0801		126 5861		928 1016
0.610	^{0}9495 6485		^{0}4589 2655		^{0}5520 9478	
		−152 7935		−124 9731		−796 7642
1	9342 8550		4464 2924		4724 1836	
		158 1054		123 2936		683 6681
2	9184 7496		4340 9988		4040 5155	
		163 0116		121 5525		586 3324
3	9021 7380		4219 4463		3454 1831	
		167 5103		119 7543		502 6056
4	8854 2277		4099 6920		2951 5775	
		171 6005		117 9042		430 6235
0.615	^{0}8682 6272		^{0}3981 7878		^{0}2520 9540	
		−175 2834		−116 0066		−368 7710
6	8507 3438		3865 7812		2152 1830	
		178 5608		114 0658		315 6503
7	8328 7830		3751 7154		1836 5327	
		181 4364		112 0867		270 0524
8	8147 3466		3639 6287		1566 4803	
		183 9153		110 0734		230 9320
9	7963 4313		3529 5553		1335 5483	
		186 0032		108 0299		197 3857
0.620	^{0}7777 4281		^{0}3421 5254		^{1}1138 1626	
		−187 7076		−105 9607		−1686 3406
1	7589 7205		3315 5647		^{0}9695 2854	
		189 0367		103 8694		1440 0388
2	7400 6838		3211 6953		8255 2466	
		189 9996		101 7601		1229 1482
3	7210 6842		3109 9352		7026 0984	
		190 6069		99 6361		1048 6659
4	7020 0773		3010 2991		5977 4325	
		190 8694		97 5015		894 2824
0.625	^{0}6829 2079		^{0}2912 7976		^{0}5083 1501	
		−190 7987		−95 3600		−762 2869
6	6638 4092		2817 4376		4320 8632	
		190 4075		93 2216		649 4865
7	6448 0017		2724 2160		3671 3767	
		189 7089		91 1174		553 1355
8	6258 2928		2633 0986		3118 2412	
		188 7162		89 1088		470 8733
9	6069 5766		2543 9898		2647 3679	
		187 4436		87 2713		400 6725

TABLE II—*Continued*

λ (micron)	$\bar{x}(\lambda)$	$\Delta\bar{x}$	$\bar{y}(\lambda)$	$\Delta\bar{y}$	$\bar{z}(\lambda)$	$\Delta\bar{z}$
0.630	^{9}5882 1330		^{9}2456 7185		^{7}2246 6954	
		−185 9051		−85 6585		−340 7921
1	5696 2279		2371 0600		1905 9033	
		184 1157		84 2755		289 7381
2	5512 1122		2286 7845		1616 1652	
		182 0898		83 0781		246 2291
3	5330 0224		2203 7064		1369 9361	
		179 8426		81 9901		209 1667
4	5150 1798		2121 7163		1160 7694	
		177 3889		80 9255		1776 0959
0.635	^{8}4972 7909		^{9}2040 7908		^{8}9831 5983	
		−174 7438		−79 8069		−1507 5187
6	4798 0471		1960 9839		8324 0796	
		171 9220		78 5756		1279 0366
7	4626 1251		1882 4083		7045 0430	
		168 9386		77 1956		1084 7476
8	4457 1865		1805 2127		5960 2954	
		165 8078		75 6513		919 6050
9	4291 3787		1729 5614		5040 6904	
		162 5442		73 9443		779 2962
0.640	^{8}4128 8345		^{9}1655 6171		^{8}4261 3942	
		−159 1620		−72 0877		−660 1364
1	3969 6725		1583 5294		3601 2578	
		155 6747		70 1030		558 9799
2	3813 9978		1513 4264		3042 2779	
		152 0958		68 0146		473 1421
3	3661 9020		1445 4118		2569 1358	
		148 4385		65 8490		400 3331
4	3513 4635		1379 5628		2168 8027	
		144 7153		63 6313		338 6001
0.645	^{8}3368 7482		^{9}1315 9315		^{8}1830 2026	
		−140 9382		−61 3846		−286 2792
6	3227 8100		1254 5469		1543 9234	
		137 1191		59 1297		241 9531
7	3090 6909		1195 4172		1301 9703	
		133 2690		56 8840		204 4150
8	2957 4219		1138 5332		1097 5553	
		129 3987		54 6625		1726 3767
9	2828 0232		1083 8707		^{9}9249 1766	
		125 5183		52 4769		1457 4753
0.650	^{8}2702 5049		^{9}1031 3938		^{7}7791 7013	
		−121 6375		−503 3674		−1230 0168
1	2580 8674		^{1}9810 5706		6561 6845	
		117 7654		482 4922		1037 6871
2	2463 1020		9328 0784		5523 9974	
		113 9102		462 1970		875 1220
3	2349 1918		8865 8814		4648 8754	
		110 0804		442 5187		737 7667
4	2239 1114		8423 3627		3911 1087	
		106 2832		423 4818		621 7545

TABLE II—*Continued*

λ (micron)	$\bar{x}(\lambda)$	$\Delta\bar{x}$	$\bar{y}(\lambda)$	$\Delta\bar{y}$	$\bar{z}(\lambda)$	$\Delta\bar{z}$
0.655	^{2}2132 8282		^{1}7999 8809		^{2}3289 3542	
		−102 5255		−405 0993		−523 8052
6	2030 3027		7594 7816		2765 5490	
		98 8137		387 3761		441 1363
7	1931 4890		7207 4055		2324 4127	
		95 1539		370 3099		371 3890
8	1836 3351		6837 0956		1953 0237	
		91 5510		353 8933		312 5649
9	1744 7841		6483 2023		1640 4588	
		88 0104		338 1148		262 9707
0.660	^{2}1656 7737		^{1}6145 0875		^{2}1377 4881	
		−84 5361		−322 9599		
1	1572 2376		5822 1276			
		81 1321		308 4119		
2	1491 1055		5513 7157			
		77 8018		294 4530		
3	1413 3037		5219 2627			
		74 5485		281 0639		
4	1338 7552		4938 1988			
		71 3744		268 2253		
0.665	^{2}1267 3808		^{1}4669 9735			
		−68 2818		−255 9172		
6	1199 0990		4414 0563			
		65 2727		244 1204		
7	1133 8263		4169 9359			
		62 3483		232 8152		
8	1071 4780		3937 1207			
		59 5098		221 9829		
9	1011 9682		3715 1378			
		567 5805		211 6049		
0.670	^{1}9552 1015		^{1}3503 5329			
		−540 9332		−198 4042		
1	9011 1683		3305 1287			
		515 1591		188 9508		
2	8496 0092		3116 1779			
		490 2572		179 8172		
3	8005 7520		2936 3607			
		466 2241		171 0023		
4	7539 5279		2765 3584			
		443 0542		162 5040		
0.675	^{1}7096 4737		^{1}2602 8544			
		−420 7397		−154 3195		
6	6675 7340		2448 5349			
		399 2709		146 4451		
7	6276 4631		2302 0898			
		378 6365		138 8769		
8	5897 8266		2163 2129			
		358 8236		131 6098		
9	5539 0030		2031 6031			
		339 8176		124 6388		

TABLE II—*Continued*

λ (micron)	$\bar{x}(\lambda)$	$\Delta\bar{x}$	$\bar{y}(\lambda)$	$\Delta\bar{y}$	$\bar{z}(\lambda)$	$\Delta\bar{z}$
0.680	^{1}5199 1854		^{1}1906 9643			
		−321 6031		−117 9580		
1	4877 5823		1789 0063			
		304 1632		111 5614		
2	4563 4191		1677 4449			
		287 4801		105 4423		
3	4285 9390		1572 0026			
		271 5356		99 5942		
4	4014 4034		1472 4084			
		256 3101		94 0098		
0.685	^{1}3758 0933		^{1}1378 3986			
		−241 7840		−88 6818		
6	3516 3093		1289 7168			
		227 9369		83 6030		
7	3288 3724		1206 1138			
		214 7482		78 7657		
8	3073 6242		1127 3481			
		202 1970		74 1621		
9	2871 4272		1053 1860			
		190 2622		697 8463		
0.690	^{1}2681 1650		^{2}9834 0137			
		−178 9228		−656 2553		
1	2502 2422		9177 7584			
		168 1573		616 7697		
2	2334 0849		8560 9887			
		157 9450		579 3127		
3	2176 1399		7981 6760			
		148 2646		543 8072		
4	2027 8753		7437 8688			
		139 0957		510 1772		
0.695	^{1}1888 7796		^{2}6927 6916			
		−130 4176		−478 3474		
6	1758 3620		6449 3442			
		122 2101		448 2438		
7	1636 1519		6001 1004			
		114 4534		419 7936		
8	1521 6985		5581 3068			
		107 1279		392 9254		
9	1414 5706		5188 3814			
		100 2148		367 5692		
0.700	^{1}1314 3558		^{2}4820 8122			
		−93 6953		−343 6568		
1	1220 6605		4477 1554			
		87 5512		321 1217		
2	1133 1093		4156 0337			
		81 7651		299 8990		
3	1051 3442		3856 1347			
		763 1958		279 9260		
4	^{2}9750 2462		3576 2087			
		711 9817		261 1417		

TABLE II—*Continued*

λ (micron)	$\bar{x}(\lambda)$	$\Delta\bar{x}$	$\bar{y}(\lambda)$	$\Delta\bar{y}$	$\bar{z}(\lambda)$	$\Delta\bar{z}$
0.705	^{3}9038 2645		^{3}3315 0670			
		−663 8474		−243 4868		
6	8374 4171		3071 5802			
		618 6374		226 9047		
7	7755 7797		2844 6755			
		576 2013		211 3399		
8	7179 5784		2633 3356			
		536 3947		196 7396		
9	6643 1837		2436 5960			
		499 0783		183 0526		
0.710	^{3}6144 1054		^{3}2253 5434			
		−464 1181		−170 2299		
1	5679 9873		2083 3135			
		431 3858		158 2242		
2	5248 6015		1925 0893			
		400 7581		146 9906		
3	4847 8434		1778 0987			
		372 1171		136 4857		
4	4475 7263		1641 6130			
		345 3501		126 6679		
0.715	^{3}4130 3762		^{3}1514 9451			
		−320 3493		−117 4982		
6	3810 0269		1397 4469			
		297 0120		108 9385		
7	3513 0149		1288 5084			
		275 2401		100 9529		
8	3237 7748		1187 5555			
		254 9403		93 5074		
9	2982 8345		1094 0481			
		236 0239		86 5692		
0.720	^{3}2746 8106		^{3}1007 4789			
		−218 4068		−801 0749		
1	2528 4038		^{3}9273 7141			
		202 0087		740 9301		
2	2326 3951		8532 7840			
		186 7537		684 9780		
3	2139 6414		7847 8060			
		172 5702		632 9551		
4	1967 0712		7214 8509			
		159 3898		584 6121		
0.725	^{3}1807 6814		^{3}6630 2388			
		−147 1483		−539 7126		
6	1660 5331		6090 5262			
		135 7849		498 0339		
7	1524 7482		5592 4923			
		125 2423		459 3652		
8	1399 5059		5133 1271			
		115 4661		423 5082		
9	1284 0398		4709 6189			
		106 4054		390 2754		

TABLE II—*Continued*

λ (micron)	$\bar{x}(\lambda)$	$\Delta\bar{x}$	$\bar{y}(\lambda)$	$\Delta\bar{y}$	$\bar{z}(\lambda)$	$\Delta\bar{z}$
0.730	1177 6344		4319 3435			
		−98 0124		−359 4909		
1	1079 6220		3959 8526			
		902 4140		330 9888		
2	9893 8060		3628 8638			
		830 5034		304 6132		
3	9063 3026		3324 2506			
		763 9911		280 2177		
4	8299 3115		3044 0329			
		702 5025		257 6649		
0.735	7596 8090		2786 3680			
		−645 6865		−236 8257		
6	6951 1225		2549 5423			
		593 2133		217 5796		
7	6357 9092		2331 9627			
		544 7748		199 8133		
8	5813 1344		2132 1494			
		500 0819		183 4207		
9	5313 0525		1948 7287			
		458 8648		168 3031		
0.740	4854 1877		1780 4256			
		−420 8711		−154 3676		
1	4433 3166		1626 0580			
		385 8649		141 5281		
2	4047 4517		1484 5299			
		353 6266		129 7036		
3	3693 8251		1354 8263			
		323 9506		118 8191		
4	3369 8745		1236 0072			
		296 6462		108 8043		
0.745	3073 2283		1127 2029			
		−271 5347		−99 5939		
6	2801 6936		1027 6090			
		248 4508		911 2711		
7	2553 2428		9364 8189			
		227 2400		833 4738		
8	2326 0028		8531 3451			
		207 7589		762 0210		
9	2118 2439		7769 3241			
		189 8744		696 4239		
0.750	1928 3695		7072 9002			
		−173 4627		−636 2288		
1	1754 9068		6436 6714			
		158 4090		581 0148		
2	1596 4978		5855 6566			
		144 6069		530 3911		
3	1451 8909		5325 2655			
		131 9576		483 9060		
4	1319 9333		4841 2695			
		120 3698		441 4940		

TABLE II—*Continued*

λ (micron)	$\bar{x}(\lambda)$	$\Delta\bar{x}$	$\bar{y}(\lambda)$	$\Delta\bar{y}$	$\bar{z}(\lambda)$	$\Delta\bar{z}$
0.755	^{3}1199 5635		^{4}4399 7755			
		−109 7588		−402 5748		
6	1089 8047		3997 2007			
		1000 4627		366 9512		
7	^{4}9897 5843		3630 2495			
		911 5982		334 3572		
8	8985 9861		3295 8923			
		830 3247		304 5476		
9	8155 6614		2991 3447			
		756 0240		277 2955		
0.760	^{4}7399 6374		^{4}2714 0492			
		−688 1251		−252 3915		
1	6711 5123		2461 6577			
		626 1011		229 6422		
2	6085 4112		2232 0155			
		569 4660		208 8695		
3	5515 9452		2023 1460			
		517 7720		189 9092		
4	4998 1732		1833 2368			
		470 6063		172 6096		
0.765	^{4}4527 5669		^{4}1660 6272			
		−427 5888		−156 8316		
6	4099 9781		1503 7956			
		388 3697		142 4468		
7	3711 6084		1361 3488			
		352 6274		129 3371		
8	3358 9810		1232 0117			
		320 0657		117 3942		
9	3038 9153		1114 6175			
		290 4128		106 5180		
0.770	^{4}2748 5025		^{4}1008 0995			
		−263 4188		−966 1710		
1	2485 0837		^{5}9114 8240			
		238 8546		876 0739		
2	2246 2291		8238 7501			
		216 5093		794 1159		
3	2029 7198		7444 6342			
		196 1901		719 5888		
4	1833 5297		6725 0454			
		177 7199		651 8436		
0.775	^{4}1655 8098		^{4}6073 2018			
		−160 9365		−590 2850		
6	1494 8733		5482 9168			
		145 6912		534 3680		
7	1349 1821		4948 5488			
		131 8478		483 5935		
8	1217 3343		4464 9553			
		119 2821		437 5045		
9	1098 0522		4027 4508			
		1078 7994		395 6834		

TABLE II—*Continued*

λ (micron)	$\bar{x}(\lambda)$	$\Delta\bar{x}$	$\bar{y}(\lambda)$	$\Delta\bar{y}$	$\bar{z}(\lambda)$	$\Delta\bar{z}$
0.780	$^{5}9901\ 7226$		$^{5}3631\ 7674$			
		$-975\ 3712$		$-357\ 7480$		
1	8926 3514		3274 0194			
		881 5850		323 3489		
2	8044 7664		2950 6705			
		796 5709		292 1674		
3	7248 1955		2658 5031			
		719 5341		263 9117		
4	6528 6614		2394 5914			
		649 7496		238 3161		
0.785	$^{5}5878\ 9118$		$^{5}2156\ 2753$			
		$-586\ 5557$		$-215\ 1377$		
6	5292 3561		1941 1376			
		529 3487		194 1552		
7	4763 0074		1746 9824			
		477 5786		175 1669		
8	4285 4288		1571 8155			
		430 7437		157 9888		
9	3854 6851		1413 8267			
		388 3875		142 4533		
0.790	$^{5}3466\ 2976$		$^{5}1271\ 3734$			
		$-350\ 0936$		$-128\ 4078$		
1	3116 2040		1142 9656			
		315 4839		115 7136		
2	2800 7201		1027 2520			
		284 2134		1042 4421		
3	2516 5067		$^{5}9230\ 0779$			
		255 9690		938 8468		
4	2260 5377		8291 2311			
		230 4659		845 3057		
0.795	$^{5}2030\ 0718$		$^{5}7445\ 9254$			
		$-207\ 4449$		$-760\ 8603$		
6	1822 6269		6685 0561			
		186 6710		684 6745		
7	1635 9559		6000 3816			
		167 9306		615 9381		
8	1468 0253		5384 4435			
		151 0296		553 9485		
9	1316 9957		4830 4950			
		135 7922		498 0605		
0.800	$^{5}1181\ 2035$		$^{5}4332\ 4345$			
		$-122\ 0586$		$-447\ 6885$		
1	1059 1449		3884 7460			
		1096 8433		402 3015		
2	$^{6}9494\ 6057$		3482 4445			
		985 3779		361 4183		
3	8509 2278		3121 0262			
		885 0031		324 6026		
4	7624 2247		2796 4236			
		794 6409		291 4595		

TABLE II—*Continued*

λ (micron)	$\bar{x}(\lambda)$	$\Delta\bar{x}$	$\bar{y}(\lambda)$	$\Delta\bar{y}$	$\bar{z}(\lambda)$	$\Delta\bar{z}$
0.805	^{6}6829 5838		^{6}2504 9641			
		−713 3162		−261 6311		
6	6116 2676		2243 3330			
		640 1461		234 7937		
7	5476 1215		2008 5393			
		574 3315		210 6540		
8	4901 7900		1797 8853			
		515 1497		188 9474		
9	4386 6403		1608 9379			
		461 9472		169 4336		
0.810	^{6}3924 6931		^{6}1439 5043			
		−414 1331		−151 8963		
1	3510 5600		1287 6080			
		371 1736		136 1396		
2	3139 3864		1151 4684			
		332 5865		121 9865		
3	2806 7999		1029 4819			
		297 9359		1088 4113		
4	2508 8640		^{7}9202 0457			
		266 8291		978 6791		
0.815	^{6}2242 0349		^{7}8223 3666			
		−238 9106		−876 2800		
6	2003 1243		7347 0866			
		213 8609		784 4016		
7	1789 2634		6562 6850			
		191 3906		701 9853		
8	1597 8728		5860 6997			
		171 2399		628 0758		
9	1426 6329		5232 6239			
		153 1736		561 8125		
0.820	^{6}1273 4593		^{7}4670 8114			
		−136 9807		−502 4196		
1	1136 4786		4168 3918			
		122 4704		449 1986		
2	1014 0082		3719 1932			
		1094 7131		401 5204		
3	^{7}9045 3689		3317 6728			
		978 2897		358 8184		
4	8067 0792		2958 8544			
		874 0439		320 5831		
0.825	^{7}7193 0353		^{7}2638 2713			
		−780 7256		−286 3556		
6	6412 3097		2351 9157			
		697 2099		255 7236		
7	5715 0998		2096 1921			
		622 4857		228 3161		
8	5092 6141		1867 8760			
		555 6438		203 7998		
9	4536 9703		1664 0762			
		495 8674		181 8749		

TABLE II—*Continued*

λ (micron)	$\bar{x}(\lambda)$	$\Delta\bar{x}$	$\bar{y}(\lambda)$	$\Delta\bar{y}$	$\bar{z}(\lambda)$	$\Delta\bar{z}$
0.830	^{7}4041 1029		^{7}1482 2013			
		−442 4226		−162 2724		
1	3598 6803		1319 9289			
		394 6502		144 7503		
2	3204 0301		1175 1786			
		351 9583		129 0918		
3	2852 0718		1046 0868			
		313 8156		1151 0171		
4	2538 2562		^{8}9309 8509			
		279 7456		1026 0546		
0.835	^{7}2258 5106		^{8}8283 7963			
		−249 3203		−914 4604		
6	2009 1903		7369 3359			
		222 1562		814 8276		
7	1787 0341		6554 5083			
		197 9093		725 8946		
8	1589 1248		5828 6137			
		176 2714		646 5307		
9	1412 8534		5182 0830			
		156 9661		575 7222		
0.840	^{7}1255 8873		^{8}4606 3608			
		−139 7456		−512 5611		
1	1116 1417		4093 7997			
		1243 8863		456 2342		
2	^{8}9917 5307		3637 5655			
		1106 9623		406 0131		
3	8810 5684		3231 5524			
		984 9083		361 2460		
4	7825 6601		2870 3064			
		876 1329		321 3493		
0.845	^{8}6949 5272		^{9}2548 9571			
		−779 2129		−285 8007		
6	6170 3143		2263 1564			
		692 8745		254 1335		
7	5477 4398		2009 0229			
		615 9794		225 9298		
8	4861 4604		1783 0931			
		547 5092		200 8161		
9	4313 9512		1582 2770			
		486 5536		178 4588		
0.850	^{8}3827 3976		^{9}1403 8182			
		−432 2996		−158 5595		
1	3395 0980		1245 2587			
		384 0203		140 8515		
2	3011 0777		1104 4072			
		341 0669		1250 9697		
3	2670 0108		^{9}9793 1023			
		302 8595		1110 8324		
4	2367 1513		8682 2699			
		268 8810		986 2052		

TABLE II—*Continued*

λ (micron)	$\bar{x}(\lambda)$	$\Delta\bar{x}$	$\bar{y}(\lambda)$	$\Delta\bar{y}$	$\bar{z}(\lambda)$	$\Delta\bar{z}$
0.855	^{9}2098 2703		^{9}7696 0647			
		−238 6691		−875 3940		
6	1859 6012		6820 6707			
		211 8121		776 8872		
7	1647 7891		6043 7835			
		187 9419		689 3361		
8	1459 8472		5354 4474			
		166 7309		611 5376		
9	1293 1163		4742 9098			
		147 8863		542 4193		
0.860	^{9}1145 2300		^{9}4200 4905			
		−131 1475		−481 0247		
1	1014 0825		3719 4658			
		1162 8214		426 5013		
2	^{8}8978 0036		3292 9645			
		1030 8308		378 0895		
3	7947 1728		2914 8750			
		913 6577		335 1127		
4	7033 5151		2579 7623			
		809 6589		296 9678		
0.865	^{8}6223 8562		^{9}2282 7945			
		−717 3705		−263 1181		
6	5506 4857		2019 6764			
		635 4894		233 0857		
7	4870 9963		1786 5907			
		562 8557		206 4450		
8	4308 1406		1580 1457			
		498 4370		182 8174		
9	3809 7036		1397 3283			
		441 3147		161 8660		
0.870	^{8}3368 3889		^{9}1235 4623			
		−390 6716		−143 2911		
1	2977 7173		1092 1712			
		345 7812		1268 2608		
2	2631 9361		^{10}9653 4512			
		305 9971		1122 3404		
3	2325 9390		8531 1108			
		270 7449		993 0416		
4	2055 1941		7538 0692			
		239 5137		878 4917		
0.875	^{9}1815 6804		^{10}6659 5775			
		−211 8501		−777 0266		
6	1603 8303		5882 5509			
		187 3506		687 1673		
7	1416 4797		5195 3836			
		165 6573		607 6002		
8	1250 8224		4587 7834			
		146 4521		537 1588		
9	1104 3703		4050 6246			
		1294 5238		474 8073		

TABLE II—*Continued*

λ (micron)	$\bar{x}(\lambda)$	$\Delta\bar{x}$	$\bar{y}(\lambda)$	$\Delta\bar{y}$	$\bar{z}(\lambda)$	$\Delta\bar{z}$
0.880	^{10}9749 1792		^{10}3575 8173			
		−1144 0764		−419 6259		
1	8605 1028		3156 1914			
		1010 9525		370 7985		
2	7594 1503		2785 3929			
		893 1774		327 6009		
3	6700 9729		2457 7920			
		788 9089		289 3901		
4	5911 9740		2168 4019			
		696 8628		255 5962		
0.885	^{10}5215 1112		^{10}1912 8057			
		−615 3904		−225 7138		
6	4599 7208		1687 0919			
		543 3596		199 2942		
7	4056 3612		1487 7977			
		479 6865		175 9400		
8	3576 6747		1311 8577			
		423 4104		155 2991		
9	3153 2643		1156 5586			
		373 6803		137 0590		
0.890	^{10}2779 5840		^{10}1019 4996			
		−329 7415		−1209 4299		
1	2449 8425		^{11}8985 5661			
		290 9259		1067 0620		
2	2158 9166		7918 5041			
		256 6415		941 3134		
3	1902 2751		6977 1907			
		226 3642		830 2016		
4	1675 9109		6146 9291			
		199 6297		732 2046		
0.895	^{10}1476 2812		^{11}5414 7245			
		−176 0271		−645 6347		
6	1300 2541		4769 0898			
		155 1929		569 2184		
7	1145 0612		4199 8714			
		136 8048		501 7749		
8	1008 2564		3698 0965			
		1205 7856		442 2595		
9	^{11}8876 7784		3255 8370			
		1062 6177		389 7484		
0.900	^{11}7814 1607		^{11}2866 0886			

Note: The superscripts refer to the number of zeros between the given number and the decimal point.

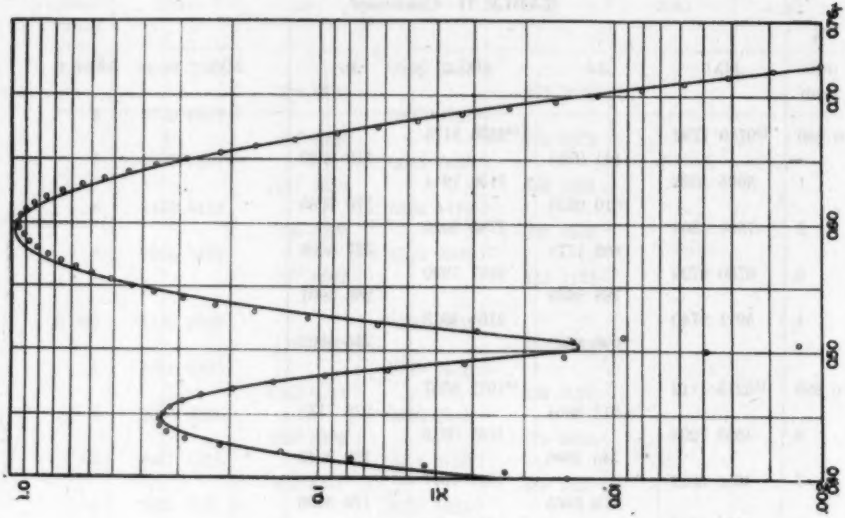

FIG. 3. Comparison of calculated and C.I.E. values for $\bar{z}$.

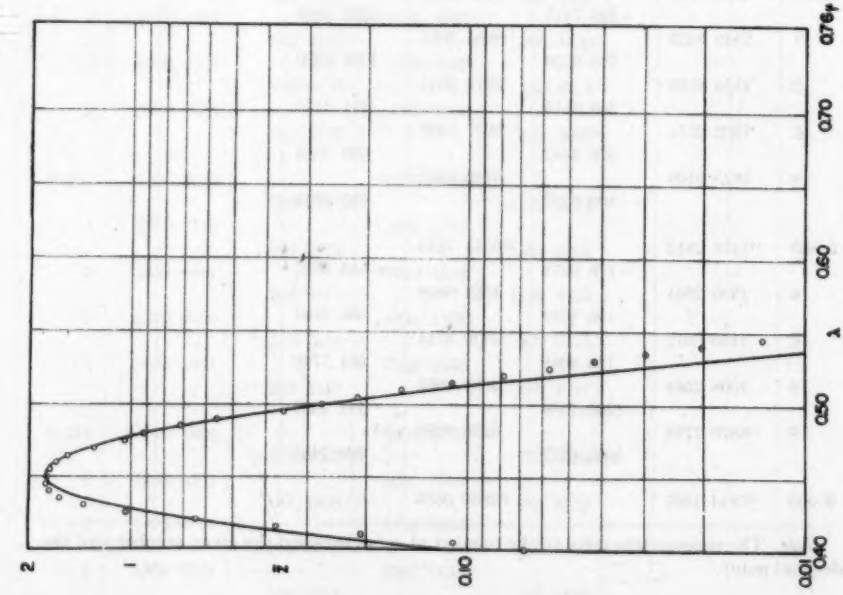

FIG. 2. Comparison of calculated and C.I.E. values for $\bar{z}$. Curve is calculated, points represent the standard tabulated values.

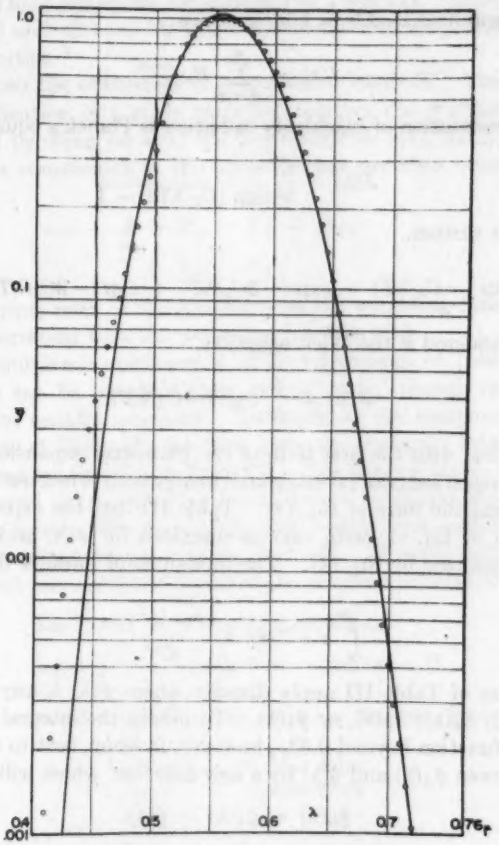

FIG. 4. Comparison of calculated and C.I.E. values for $\bar{y}$

3. INTEGRATION

One of the principal advantages of the new functions is that they allow analytic integrations. In practice, these integrations may be divided into two principal classes, giving

(a) Specification of radiation from a source,

(b) Specification of radiation reflected from a surface or transmitted through a medium.

The *radiation from a source* may be represented in some cases by a power series:

$$J(\lambda) = \sum_{m=0}^{\infty} K_m \lambda^m . \tag{10}$$

Or it may be approximated by a Fourier series:

$$J(\lambda) = \sum_{m=-\infty}^{\infty} B_m e^{im/\lambda}. \tag{11}$$

The best representation of blackbody radiation is Planck's equation,

$$J(\lambda) = \frac{C_1}{\lambda^5} \frac{1}{\exp(C_2/\lambda T) - 1} \tag{12}$$

which may be written,

$$J(\lambda) = \frac{C_1}{\lambda^5}\left[\exp(-C_2/\lambda T) + \exp(-2C_2/\lambda T) + \exp(-3C_2/\lambda T) + \cdots\right]. \tag{13}$$

Also to be considered is the Wien equation,

$$J(\lambda) = \frac{C_1}{\lambda^5} \exp(-C_2/\lambda T), \tag{14}$$

which is identical with the first term of the Planckian expansion, Eq. (13).

All of these equations can be integrated analytically when used with weighting functions having the form of Eq. (4). Table III lists the expressions obtained by integration of Eq. (1) with various functions for $J(\lambda)$ and with weighting functions represented by Eq. (4). The fundamental formula that is used in all cases is

$$\int_0^\infty x^r e^{-ax} dx = \frac{\Gamma(r+1)}{a^{r+1}}. \tag{15}$$

The formulas of Table III apply directly where $w(\lambda)$ is any one of the four functions: $\bar{z}(\lambda)$, $\bar{x}_B(\lambda)$, $\bar{x}_C(\lambda)$, or $\bar{y}_A(\lambda)$. To obtain the integral for the peculiar part of the $\bar{y}$-function beyond 0.63μ, however, it seems best to approximate the difference between $\bar{y}_A(\lambda)$ and $\bar{y}(\lambda)$ by a new function[5] which will be called $\bar{y}_D(\lambda)$:

$$\bar{y}_D(\lambda) = \bar{y}_A(\lambda) - \bar{y}(\lambda). \tag{16}$$

The difference is found to be closely approximated[5] by a function of the form of Eq. (4). The constants are given in Table I.

If the *radiation is reflected* from a selectively absorbing surface having the spectral reflectance $\rho(\lambda)$, Eq. (3) becomes

$$\begin{aligned} X &= \int_0^\infty \rho(\lambda)\bar{x}(\lambda)J(\lambda)d\lambda, \\ Y &= \int_0^\infty \rho(\lambda)\bar{y}(\lambda)J(\lambda)d\lambda, \\ Z &= \int_0^\infty \rho(\lambda)\bar{z}(\lambda)J(\lambda)d\lambda. \end{aligned} \tag{17}$$

The trichromatic specification of radiation transmitted through a material having the spectral transmittance $\tau(\lambda)$ is obtained from Eq. (17) by substituting

$\tau(\lambda)$ for $\rho(\lambda)$. These conditions are included in Table III. The functions used to represent $\rho(\lambda)$ and $\tau(\lambda)$ are the power series, the Fourier series and the Pearson Type *V* function.[6]

Table III allows the evaluation of colorimetric integrals. The complete trichromatic specification of a given radiation requires the evaluation of five integrals obtained by using for $w(\lambda)$ the five functions: $\bar{z}(\lambda)$, $\bar{x}_B(\lambda)$, $\bar{x}_C(\lambda)$, $\bar{y}_A(\lambda)$, and $\bar{y}_D(\lambda)$. The coordinates in the affine 3-space are then obtained from the relations:

$$\begin{aligned} X &= X_A + X_C - X_B, \\ Y &= Y_A - Y_D, \end{aligned} \tag{18}$$

where the subscripts refer to the subscripts on the weighting functions and Z is obtained by integration with the weighting function $\bar{z}(\lambda)$.

The gamma function is employed in all the expressions of Table III. Values of this function can be obtained from Peters' table,[6] though rather laborious interpolations are usually necessary. To minimize the computations required in the routine use of Table III, we have made the necessary interpolations and have listed the gamma function in Table IV. Ten decimal places were employed in the interpolation, values being read from Peters' table and Newton's formula being used with fourth differences. The final values were rounded off to 8 figures. The table runs from $(p - 5)$ to $(p + 10)$. If other values are needed, they can be found very easily by means of the recurrence formulas,[6]

$$\begin{aligned} \log \Gamma(x + 1) &= \log \Gamma(x) + \log x, \\ \log \Gamma(x - 1) &= \log \Gamma(x) - \log (x - 1). \end{aligned} \tag{19}$$

Example 1.

Given $\log \Gamma(p) = 776.70542\ 38193$ for $\dot{z}$, for which $p = 365.33880$ (Table I). What is $\log \Gamma(p + 1)$?

$$\begin{aligned} \log \Gamma(p) &= 776.70542\ 38193 \\ \log p &= 2.56269\ 57979 \end{aligned}$$

By Eq. (19),

$$\log \Gamma(p + 1) = 779.26811\ 96172.$$

Example 2.

Determine the trichromatic coordinates X, Y, Z for the radiation:

$$J(\lambda) = K_2\lambda^2.$$

According to Table III,

$$\log (L/K_2) = \log A + \log \Gamma(p - 3) - (p - 3) \log q.$$

For Z, by Tables I and IV,

$$\begin{aligned} \log A &= 32.75260, \\ p &= 365.33880, \\ \log q &= 2.21735\ 38996, \\ \log \Gamma(p - 3) &= 769.02449\ 178. \end{aligned}$$

TABLE III

Integrals

$$L = \int_0^\infty w(\lambda)J(\lambda)d\lambda, \qquad w(\lambda) = \rho(\lambda)(A/\lambda^p)\exp(-q/\lambda) \text{ or } w(\lambda) = r(\lambda)(A/\lambda^p)\exp(-q/\lambda).$$

Condition	$J(\lambda)$	$\rho(\lambda)$ or $r(\lambda)$	L	$\log L$
(I) Equal-energy spectrum	K	1	$AK\dfrac{\Gamma(p-1)}{q^{p-1}}$	$\log A + \log K + \log\Gamma(p-1) - (p-1)\log q$
(II) Power function	$K_m\lambda^m$	1	$AK_m\dfrac{\Gamma(p-m-1)}{q^{p-m-1}}$	$\log A + \log K_m + \log\Gamma(p-m-1) - (p-m-1)\log q$
(III) Trigonometric function	$B_m\exp(im/\lambda)$	1	$AB_m\dfrac{\Gamma(p-1)}{(q-im)^{p-1}}$	$\log A + \log B_m + \log\Gamma(p-1) - (p-1)\log(q-im)$
(IV) Generalized Wien function	$\dfrac{C_1}{\lambda^5}\exp\left(\dfrac{-nC_2}{\lambda T}\right)$	1	$AC_1\dfrac{\Gamma(p+4)}{(q+nC_2/T)^{p+4}}$	$\log A + \log C_1 + \log\Gamma(p+4) - (p+4)\log(q+nC_2/T)$
(V) Rayleigh-Jeans function	$\dfrac{C_1T}{C_2\lambda^4}[1 + y/(2)! + y^2/(3)! + \cdots]$ where $y = C_2/\lambda T$	1	$\dfrac{AC_1T}{C_2}\dfrac{\Gamma(p+3)}{q^{p+3}}\left[1 - \dfrac{1}{2}\left(\dfrac{C_2}{T}\right)\dfrac{p+4}{q} + \dfrac{1}{12}\left(\dfrac{C_2}{T}\right)^2\dfrac{(p+4)(p+5)}{q^2} - \cdots\right]$	
Limiting case, $T \to \infty$	$C_1T/(C_2\lambda^4)$	1	$\dfrac{AC_1T}{C_2}\dfrac{\Gamma(p+3)}{q^{p+3}}$	$\log(AC_1/C_2) + \log T + \log\Gamma(p+3) - (p+3)\log q$
(VI) Reflected radiation, power function	$\dfrac{C_1}{\lambda^5}\exp(-nC_2/\lambda T)$	$K_m\lambda^m$	$AC_1K_m\dfrac{\Gamma(p+4-m)}{(q+nC_2/T)^{p+4-m}}$	$\log A + \log C_1 + \log K_m + \log\Gamma(p+4-m) - (p+4-m)\log(q+nC_2/T)$
(VII) Reflected radiation, trigonometric function	$\dfrac{C_1}{\lambda^5}\exp(-nC_2/\lambda T)$	$B_m\exp(im/\lambda)$	$AC_1B_m\dfrac{\Gamma(p+4)}{(q+nC_2/T-im)^{p+4}}$	$\log A + \log C_1 + \log B_m + \log\Gamma(p+4) - (p+4)\log(q+nC_2/T-im)$

(VIII) Power series	$\sum_{m=0}^{\infty} K_m \lambda^m$	1	$A \sum_{m=0}^{\infty} K_m \frac{\Gamma(p-m-1)}{q^{p-m-1}}$	
(IX) Trigonometric series	$\sum_{m=-\infty}^{\infty} B_m \exp(im/\lambda)$	1	$A \sum_{m=-\infty}^{\infty} B_m \frac{\Gamma(p-1)}{(q-im)^{p-1}}$	
(X) Planck function	$\frac{C_1}{\lambda^5} \frac{1}{\exp(C_2/\lambda T)-1}$	1	$AC_1\Gamma(p+4) \sum_{n=1}^{\infty} (q+nC_2/T)^{-(p+4)}$	
(XI) Reflected radiation, Planckian source, power series	$\frac{C_1}{\lambda^5} \frac{1}{\exp(C_2/\lambda T)-1}$	$\sum_{m=0}^{\infty} K_m \lambda^m$	$AC_1 \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{K_m \Gamma(p+4-m)}{(q+nC_2/T)^{p+4-m}}$	
(XII) Reflected radiation, Planckian source, trigonometric series	$\frac{C_1}{\lambda^5} \frac{1}{\exp(C_2/\lambda T)-1}$	$\sum_{m=-\infty}^{\infty} B_m \exp(im/\lambda)$	$AC_1 \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{B_m \Gamma(p+4)}{(q+nC_2/T-im)^{p+4}}$	
(XIII) Reflected radiation, Wien source, Pearson function	$\frac{C_1}{\lambda^5} \exp(-nC_2/\lambda T)$	$\frac{A'}{\lambda^{p'}} \exp(-q'/\lambda)$	$AA'C_1 \frac{\Gamma(p+p'+4)}{(q+q'+nC_2/T)^{p+p'+4}}$	$\log A + \log A' + \log C_1 + \log \Gamma(p+p'+4) - (p+p'+4) \log (q+q'+nC_2/T)$.
(XIV) Reflected radiation, Planckian source, Pearson function	$\frac{C_1}{\lambda^5} \frac{1}{\exp(C_2/\lambda T)-1}$	$\frac{A'}{\lambda^{p'}} \exp(-q'/\lambda)$	$AA'C_1 \sum_{n=1}^{\infty} \frac{\Gamma(p+p'+4)}{(q+q'+nC_2/T)^{p+p'+4}}$	

Thus,

$$\begin{aligned} \log A &= 32.75260\ 000 \\ \log \Gamma(p-3) &= 769.02449\ 178 \\ -(p-3)\log q &= -803.43335\ 116 \\ \hline \log (Z/K_2) &= 8.34374\ 062 - 10 \\ \text{or } Z/K_2 &= 0.022066\ 864. \end{aligned}$$

In a similar manner,

$$\begin{aligned} X_A/K_2 &= 0.004828\ 635 \\ X_C/K_2 &= 0.031664\ 101 \\ -X_B/K_2 &= -0.001266\ 069 \end{aligned}$$

TABLE IV

The gamma function

	$\bar{x}(\lambda)$	$\bar{x}_B(\lambda)$	$\bar{x}_C(\lambda)$	$\bar{y}_A(\lambda)$	$\bar{y}_D(\lambda)$
p	365.3388	500.0000	336.0385	182.1905	1013.5000
$\log \Gamma(p-1)$	774.1439 1840	1128.6893 3799	699.6352 2470	329.7328 7408	2602.1357 3502
$\log \Gamma(p-2)$	771.5836 0662	1125.9921 0864	697.1114 2818	327.4771 4219	2599.1307 6913
$\log \Gamma(p-3)$	769.0244 9178	1123.2957 5225	694.5889 3374	325.2238 2721	2596.1262 3282
$\log \Gamma(p-4)$	766.4665 7718	1120.6002 7058	692.0677 4529	322.9729 4266	2593.1221 2649
$\log \Gamma(p-5)$	763.9098 6615	1117.9056 6538	689.5478 6679	320.7245 0223	2590.1184 5059
$\log \Gamma(p)$	776.7054 2382	1131.3874 3853	702.1603 1942	331.9910 0950	2605.1411 3005
$\log \Gamma(p+1)$	779.2681 1962	1134.0864 0854	704.6867 0846	334.2515 3523	2608.1469 5381
$\log \Gamma(p+2)$	781.8320 0254	1136.7862 4626	707.2143 8797	336.5144 3818	2611.1532 0586
$\log \Gamma(p+3)$	784.3970 6934	1139.4869 4998	709.7433 5414	338.7797 0541	2614.1598 8578
$\log \Gamma(p+4)$	786.9633 1681	1142.1885 1796	712.2736 0315	341.0473 2411	2617.1669 9317
$\log \Gamma(p+5)$	789.5307 4174	1144.8909 4850	714.8051 3124	343.3172 8163	2620.1745 2759
$\log \Gamma(p+6)$	792.0993 4096	1147.5942 3988	717.3379 3465	345.5895 6543	2623.1824 8862
$\log \Gamma(p+7)$	794.6691 1129	1150.2983 9039	719.8720 0965	347.8641 6313	2626.1908 7585
$\log \Gamma(p+8)$	797.2400 4958	1153.0033 9835	722.4073 5251	350.1410 6245	2629.1996 8886
$\log \Gamma(p+9)$	799.8121 5271	1155.7092 6207	724.9439 5956	352.4202 5127	2632.2089 2723
$\log \Gamma(p+10)$	802.3854 1755	1158.4159 7985	727.4818 2711	354.7017 1758	2635.2185 9055

By Eq. (18), $X/K_2 = 0.035226\ 667$

Also,

$$\begin{aligned} Y_A/K_2 &= 0.033949\ 650 \\ -Y_A/K_1 &= -0.000333\ 305 \end{aligned}$$

and by

Eq. (18), $Y/K_2 = 0.033616\ 345.$

The foregoing calculations determine the trichromatic coordinates for any value of the coefficient K_2.

Since Planckian radiation is employed as a foundation for all investigations of incandescent radiators, its practical importance is considerable. Table V lists

TABLE V

Planckian radiation

T (°K)	$1/T$ (mireds)	X (königs/m^2)	Y (youngs/m^2)	Z (priests/m^2)
1000	1000	^{1}252 948	^{1}134 395	^{0}944 996
1023	977	^{1}422 666	^{1}227 210	^{1}183 551
1125	888	^{0}322 323	^{0}182 430	^{2}254 347
1143	875	^{0}443 656	^{0}253 336	^{2}384 659
1200	833	1.15 911	^{0}680 568	^{1}133 424
1286	777	4.18 939	2.56 049	^{1}705 159
1333	750	7.98 011	4.98 249	^{0}162 570
1400	714	18.3 333	11.7 642	^{0}585 955
1421	704	23.4 134	15.1 754	^{0}656 317
1500	667	55.6 036	37.1 962	2.01 486
1556	643	97.0 509	66.3 027	4.14 918
1600	625	147. 480	102. 394	7.13 927
1750	571	519. 527	379. 277	36.5 438
1800	555	755. 356	559. 973	59.3 661
2000	500	281 2.39	220 2.97	326. 034
2046	488	367 2.88	290 9.85	460. 638
2222	450	924 4.33	761 8.78	152 0.36
2250	445	105 55.3	874 9.23	180 4.75
2333	429	154 29.9	129 97.9	294 6.99
2500	400	306 22.8	265 56.7	713 5.53
2567	375	559 24.2	497 30.3	154 96.3
2800	357	861 21.9	779 41.9	289 92.1
2842	352	978 71.1	890 30.4	318 08.2
2857	350	102 398	933 16.7	337 07.8
3000	333	153 506	142 131	566 42.9
3333	300	346 514	330 671	160 296
3500	286	492 209	475 429	250 578
4000	250	119 124^{1}	118 256^{1}	767 515
4500	222	238 695^{1}	241 170^{1}	183 810^{1}
4667	214	291 555^{1}	295 854^{1}	236 019^{1}
5000	200	418 721^{1}	427 825^{1}	370 389^{1}
6000	167	986 072^{1}	101 820^{2}	106 426^{2}
6667	150	152 435^{2}	157 840^{2}	180 949^{2}
7000	143	184 129^{2}	190 731^{2}	227 375^{2}
8000	125	297 294^{2}	307 689^{2}	403 813^{2}
9000	111	435 254^{2}	449 379^{2}	634 293^{2}
10000	100	594 759^{2}	612 190^{2}	914 511^{2}
14000	714	139 095^{3}	141 480^{3}	242 572^{3}
20000	50.0	285 807^{3}	287 271^{3}	540 902^{3}

Note: The superscripts refer to the number of zeros between the given numbers and the decimal point. For instance, at $T = 1000$, $X = {}^{1}252948 = 0.0252948$. At $T = 20000$, $X = 285807^{3} = 285{,}807{,}000$.

the trichromatic specifications of Planckian radiation at a number of temperatures. The international temperature scale is used[9] and

$$C_1 = 3.6970 \times 10^8 \text{ watt } m^{-2} \text{ micron}^4,$$

$$C_2 = 14320 \text{ micron-degree}.$$

Calculations were made by use of Eq. (12) and the integration formula (Table III):

$$\log L = \log A + \log C_1 + \log \Gamma(p+4) - (p+4) \log (q + nC_2/T). \quad (20)$$

TABLE VI

Calculation for Planckian radiation, $T = 14000°K$

$\log Z_n = 828.28376\ 626 - (p+4) \log (q + nC_2/T)$, $(p+4) = 369.33880$, $q = 164.95060$, $C_2/T = 1.0228\ 57142$

n	$(q + nC_2/T)$	$\log (q + nC_2/T)$	$(p+4) \log (q + nC_2/T)$	$\log Z_n$	Z_n
1	165.97 34571	2.22003 86400	819.94640 725	8.33735 901	2.1744 980 × 10^8
2	166.99 63143	2.22270 68859	820.93189 399	7.35187 227	2.2483 932 × 10^7
3	168.01 91714	2.22535 88386	821.91136 302	6.37240 324	2.3572 369 × 10^6
4	169.04 20286	2.22799 46957	822.88488 732	5.39887 894	2.5054 108 × 10^5
5	170.06 48857	2.23061 46513	823.85253 857	4.43122 769	2.6991 541 × 10^4
6	171.08 77429	2.23321 88967	824.81438 744	3.46937 882	2.9469 909 × 10^3
7	172.11 06000	2.23580 76186	825.77050 288	2.51326 338	326.03 437 × 1
8	173.13 34571	2.23838 10009	826.72095 282	1.56281 344	36.543 778
9	174.15 63143	2.24093 92249	827.66580 420	0.61796 206	4.1491 780
10	175.17 91714	2.24348 24677	828.60512 244	9.76786 438–10	0.58595 516
					$Z = 2.4257\ 180 \times 10^8$ priests/m^2

At low temperatures, the Wien formula, Eq. (14), is employed and results are obtained very simply. At higher temperatures, a number of components of the Wien type must be calculated. The number of terms required for 8-figure accuracy is as follows:

$T = 20{,}000°K$, 14 terms;

$T = 14{,}000$, 10 terms;

$T = 6000$, 6 terms;

$T = 3000$, 2 terms.

When a smaller number of significant figures is required, the number of terms needed in the Planck series can be greatly reduced.

Table VI is an example showing the calculation for $T = 14{,}000°K$. A helpful feature of such computations is that they apply also to other temperatures. Addition of the terms for $n = 2, 4, 6, 8, 10$ gives Z for $7000°K$. Addition of terms for $n = 3, 6, 9$ gives Z for $T = 14000/3$. In this way, the single table gives Z for 14000°, 7000°, 4667°, 3500°, 2800°, 2333°, 2000°, 1750°, etc.

Since $J(\lambda)$ specifies the radiation at the blackbody (watts per micron radiated by the blackbody per sq.m of blackbody opening), the X, Y, Z of Table V apply also to this condition. That is, they specify in absolute units the trichromatic coordinates for the Planckian radiator. No names have been standardized for these units, though we have suggested[5] the names *young/m²*, *könig/m²*, and *priest/m²** The quantity Y in Table V is expressed in youngs/m², which can be converted to lumens/m² by multiplying the tabulated value by 647.8 (See Section 2). X is a measure of the amount of red light and is expressed in königs/m², while Z is a measure of the amount of blue light and is expressed in priests/m².

4. Standard Distributions for Illuminants

In specifying the radiation reflected from a surface or transmitted through a filter, one must designate not only the spectral properties of the material but also the spectral curve of the incident radiation. In colorimetry, therefore, it has been customary to fix certain standard distribution curves[3] for the illuminant. There is advantage in having two such standard spectral distributions, one representing average artificial light and the other average daylight. We suggest[5] the two distributions:

Distribution A' = Planckian radiation at 2842°K (C_2 = 14,320).

Distribution B' = Planckian radiation at 7000°K (C_2 = 14,320).

Tabulated values for these two distributions are listed in Table VII. The computations were made at wavelength intervals of 0.001μ, using Planck's equation. Ten figures were employed, and values were checked by means of third differences. Values were then rounded off to 8 figures.

Table VII, being obtained directly from Eq. (12), is in absolute units (watt m^{-2} micron $^{-1}$). If a blackbody were used as a standard illuminant, the values of Table VII would apply only if the sample were placed within the blackbody enclosure or at its opening. When used at some distance from the blackbody, the sample would receive less radiation at all wavelengths than indicated by Table VII. The resulting spectral-distribution curve would have the same shape as that given by the table, but all ordinates would be reduced by a constant factor determined by the geometry of the set-up. When the illuminant is an incandescent lamp with filter, the absolute values of Table VII become of even less significance. On the whole, however, it seems better to tabulate absolute values than to employ the traditional method of keeping everything on a relative basis.

Example.

What is $J(\lambda)$ for Illuminant A' and with $\lambda = 0.554\mu$?

For $T = 2842°K$,

$$J(\lambda) = \left(\frac{3.6970 \times 10^{9}}{\lambda^{5}}\right)(10^{(2.1832\ 81832)/\lambda} - 1)^{-1}.$$

* See J. Opt. Soc. Am., 36, 1946, p. 120.

TABLE VII

Proposed standard distributions

$C_1 = 3.6970 \times 10^8$, $C_2 = 14320$

λ (micron)	A' ($T = 2842°K$)		B' ($T = 7000°K$)	
	$J(\lambda)$ (watt m^{-2} micron^{-1})	$\Delta J(\lambda)$	$J(\lambda)$ (watt m^{-2} micron^{-1})	$\Delta J(\lambda)$
0.360	5.102 4252 $\times 10^4$		2.088 9237 $\times 10^8$	
		+128 6538		+4 0265
1	5.231 0790		2.092 9502	
		130 9547		3 9334
2	5.362 0337		2.096 8836	
		133 2752		3 8411
3	5.495 3089		2.100 7247	
		135 6151		3 7490
4	5.630 9240		2.104 4737	
		137 9740		3 6577
0.365	5.768 8980		2.108 1314	
		140 3519		3 5668
6	5.909 2499		2.111 6982	
		142 7483		3 4764
7	6.051 9983		2.115 1746	
		145 1632		3 3867
8	6.197 1615		2.118 5613	
		147 5963		3 2974
9	6.344 7578		2.121 8587	
		150 0473		3 2088
0.370	6.494 8051		2.125 0675	
		152 5160		3 1206
1	6.647 3211		2.128 1881	
		155 0021		3 0331
2	6.802 3232		2.131 2212	
		157 5054		2 9461
3	6.959 8286		2.134 1673	
		160 0256		2 8598
4	7.119 8542		2.137 0271	
		162 5623		2 7740
0.375	7.282 4165		2.139 8011	
		165 1156		2 6887
6	7.447 5321		2.142 4898	
		167 6848		2 6042
7	7.615 2169		2.145 0940	
		170 2698		2 5201
8	7.785 4867		2.147 6141	
		172 8705		2 4367
9	7.958 3572		2.150 0508	
		175 4863		2 3538

TABLE VII—*Continued*

λ (micron)	A' (T = 2842°K)		B' (T = 7000°K)	
	$J(\lambda)$ (watt m^{-2} micron^{-1})	$\Delta J(\lambda)$	$J(\lambda)$ (watt m^{-2} micron^{-1})	$\Delta J(\lambda)$
0.380	8.133 8435		2.152 4046	
		178 1171		2 2717
1	8.311 9606		2.154 6763	
		180 7625		2 1900
2	8.492 7231		2.156 8663	
		183 4223		2 1091
3	8.676 1454		2.158 9754	
		186 0061		2 0287
4	8.862 2415		2.161 0041	
		188 7837		1 9489
0.385	9.051 0252		2.162 9530	
		191 4848		1 8698
6	9.242 5100		2.164 8228	
		194 1989		1 7912
7	9.436 7089		2.166 6140	
		196 9258		1 7133
8	9.633 6347		2.168 3273	
		199 6654		1 6361
9	9.833 3001		2.169 9634	
		202 4170		1 5594
0.390	10.03 5717		2.171 5228	
		20 5181		1 4834
1	10.24 0898		2.173 0062	
		20 7955		1 4080
2	10.44 8853		2.174 4142	
		21 0742		1 3332
3	10.65 9595		2.175 7474	
		21 3539		1 2590
4	10.87 3134		2.177 0064	
		21 6347		1 1856
0.395	11.08 9481		2.178 1920	
		21 9164		1 1127
6	11.30 8645		2.179 3047	
		22 1903		1 0404
7	11.53 0638		2.180 3451	
		22 4829		9688
8	11.75 5467		2.181 3139	
		22 7677		8978
9	11.98 3144		2.182 2117	
		23 0531		8275
0.400	12.21 3675		2.183 0392	
		23 3396		7577
1	12.44 7071		2.183 7969	
		23 6268		6887
2	12.68 3339		2.184 4856	
		23 9148		6202
3	12.92 2487		2.185 1058	
		24 2035		5523
4	13.16 4522		2.185 6581	
		24 4929		4852

TABLE VII—*Continued*

λ (micron)	A' ($T = 2842°K$) $J(\lambda)$ (watt m^{-2} micron^{-1})	$\Delta J(\lambda)$	B' ($T = 7000°K$) $J(\lambda)$ (watt m^{-2} micron^{-1})	$\Delta J(\lambda)$
0.405	13.40 9451		2.186 1433	
		24 7832		4185
6	13.65 7283		2.186 5618	
		25 0739		3527
7	13.90 8022		2.186 9145	
		25 3654		2873
8	14.16 1676		2.187 2018	
		25 6573		2226
9	14.41 8249		2.187 4244	
		25 9498		1586
0.410	14.67 7747		2.187 5830	
		26 2428		0951
1	14.94 0175		2.187 6781	
		26 5364		+0323
2	15.20 5539		2.187 7104	
		26 8302		−0299
3	15.47 3841		2.187 6805	
		27 1246		0915
4	15.74 5087		2.187 5890	
		27 4193		1524
0.415	16.01 9280		2.187 4366	
		27 7144		−2128
6	16.29 6424		2.187 2238	
		28 0096		2726
7	16.57 6520		2.186 9512	
		28 3052		3316
8	16.85 9572		2.186 6196	
		28 6011		3901
9	17.14 5583		2.186 2295	
		28 8970		4481
0.420	17.43 4553		2.185 7814	
		29 1932		−5053
1	17.72 6485		2.185 2761	
		29 4895		5621
2	18.02 1380		2.184 7140	
		29 7858		6181
3	18.31 9238		2.184 0959	
		30 0823		6736
4	18.62 0061		2.183 4223	
		30 3787		7284
0.425	18.92 3848		2.182 6939	
		30 6750		−7828
6	19.23 0598		2.181 9111	
		30 9715		8365
7	19.54 0313		2.181 0746	
		31 2677		8896
8	19.85 2990		2.180 1850	
		31 5639		9421
9	20.16 8629		2.179 2429	
		31 8598		9940

TABLE VII—*Continued*

λ (micron)	A' ($T = 2842°K$) $J(\lambda)$ (watt m^{-2} micron^{-1})	A' ($T = 2842°K$) $\Delta J(\lambda)$	B' ($T = 7000°K$) $J(\lambda)$ (watt m^{-2} micron^{-1})	B' ($T = 7000°K$) $\Delta J(\lambda)$
0.430	20.48 7227		2.178 2489	
		32 1557		−1 0454
1	20.80 8784		2.177 2035	
		32 4512		1 0962
2	21.13 3296		2.176 1073	
		32 7466		1 1463
3	21.46 0762		2.174 9610	
		33 0415		1 1960
4	21.79 1177		2.173 7650	
		33 3363		1 2450
0.435	22.12 4540		2.172 5200	
		33 6307		−1 2936
6	22.46 0847		2.171 2264	
		33 9246		1 3414
7	22.80 0093		2.169 8850	
		34 2181		1 3888
8	23.14 2274		2.168 4962	
		34 5112		1 4356
9	23.48 7386		2.167 0606	
		34 8038		1 4818
0.440	23.83 5424		2.165 5788	
		35 0958		−1 5275
1	24.18 6382		2.164 0513	
		35 3874		1 5727
2	24.54 0256		2.162 4786	
		35 6782		1 6172
3	24.89 7038		2.160 8614	
		35 9686		1 6613
4	25.25 6724		2.159 2001	
		36 2583		1 7048
0.445	25.61 9307		2.157 4953	
		36 5472		−1 7477
6	25.98 4779		2.155 7476	
		36 8355		1 7902
7	26.35 3134		2.153 9574	
		37 1230		1 8321
8	26.72 4364		2.152 1253	
		37 4097		1 8734
9	27.09 8461		2.150 2519	
		37 6957		1 9143
0.450	27.47 5418		2.148 3376	
		37 9808		−1 9547
1	27.85 5226		2.146 3829	
		38 2651		1 9944
2	28.23 7877		2.144 3885	
		38 5484		2 0338
3	28.62 3361		2.142 3547	
		38 8308		2 0725
4	29.01 1669		2.140 2822	
		39 1123		2 1108

TABLE VII—*Continued*

λ (micron)	A' ($T = 2842°K$) $J(\lambda)$ (watt m^{-2} micron^{-1})	$\Delta J(\lambda)$	B' ($T = 7000°K$) $J(\lambda)$ (watt m^{-2} micron^{-1})	$\Delta J(\lambda)$
0.455	29.40 2792		2.138 1714	
		39 3928		−2 1486
6	29.79 6720		2.136 0228	
		39 6723		2 1859
7	30.19 3443		2.133 8369	
		39 9507		2 2226
8	30.59 2950		2.131 6143	
		40 2282		2 2590
9	30.99 5232		2.129 3553	
		40 5044		2 2947
0.460	31.40 0276		2.127 0606	
		40 7796		−2 3300
1	31.80 8072		2.124 7306	
		41 0536		2 3649
2	32.21 8608		2.122 3657	
		41 3264		2 3992
3	32.63 1872		2.119 9665	
		41 5981		2 4331
4	33.04 7853		2.117 5334	
		41 8685		2 4665
0.465	33.46 6538		2.115 0669	
		42 1377		−2 4994
6	33.88 7915		2.112 5675	
		42 4055		2 5319
7	34.31 1970		2.110 0356	
		42 6721		2 5638
8	34.73 8691		2.107 4718	
		42 9374		2 5955
9	35.16 8065		2.104 8763	
		43 2013		2 6265
0.470	35.60 0078		2.102 2498	
		43 4638		−2 6572
1	36.03 4716		2.099 5926	
		43 7249		2 6873
2	36.47 1965		2.096 9053	
		43 9846		2 7171
3	36.91 1811		2.094 1882	
		44 2428		2 7464
4	37.35 4239		2.091 4418	
		44 4996		2 7753
0.475	37.79 9235		2.088 6665	
		44 7548		−2 8037
6	38.24 6783		2.085 8628	
		45 0086		2 8317
7	38.69 6869		2.083 0311	
		45 2608		2 8593
8	39.14 9477		2.080 1718	
		45 5115		2 8865
9	39.60 4592		2.077 2853	
		45 7605		2 9132

TABLE VII—*Continued*

λ (micron)	A' (T = 2842°K)		B' (T = 7000°K)	
	$J(\lambda)$ (watt m^{-2} micron^{-1})	$\Delta J(\lambda)$	$J(\lambda)$ (watt m^{-2} micron^{-1})	$\Delta J(\lambda)$
0.480	40.06 2197		2.074 3721	
		46 0080		−2 9395
1	40.52 2277		2.071 4326	
		46 2537		2 9655
2	40.98 4814		2.068 4671	
		46 4980		2 9909
3	41.44 9794		2.065 4762	
		46 7405		3 0160
4	41.91 7199		2.062 4602	
		46 9813		3 0408
0.485	42.38 7012		2.059 4194	
		47 2205		−3 0650
6	42.85 9217		2.056 3544	
		47 4578		3 0889
7	43.33 3795		2.053 2655	
		47 6935		3 1125
8	43.81 0730		2.050 1530	
		47 9274		3 1355
9	44.29 0004		2.047 0175	
		48 1595		3 1584
0.490	44.77 1599		2.043 8591	
		48 3898		−3 1806
1	45.25 5497		2.040 6785	
		48 6184		3 2027
2	45.74 1681		2.037 4758	
		48 8449		3 2242
3	46.23 0130		2.034 2516	
		49 0698		3 2455
4	46.72 0828		2.031 0061	
		49 2928		3 2663
0.495	47.21 3756		2.027 7398	
		49 5138		−3 2869
6	47.70 8894		2.024 4529	
		49 7330		3 3070
7	48.20 6224		2.021 1459	
		49 9503		3 3268
8	48.70 5727		2.017 8191	
		50 1656		3 3462
9	49.20 7383		2.014 4729	
		50 3789		3 3653
0.500	49.71 1172		2.011 1076	
		50 5904		−3 3840
1	50.21 7076		2.007 7236	
		50 7999		3 4024
2	50.72 5075		2.004 3212	
		51 0073		3 4205
3	51.23 5148		2.000 9007	
		51 2128		3 4381
4	51.74 7276		1.997 4626	
		51 4163		3 4556

TABLE VII—*Continued*

λ (micron)	A' ($T = 2842°K$) $J(\lambda)$ (watt m^{-2} micron^{-1})	$\Delta J(\lambda)$	B' ($T = 7000°K$) $J(\lambda)$ (watt m^{-2} micron^{-1})	$\Delta J(\lambda)$
0.505	52.26 1439		1.994 0070	
		51 6177		−3 4726
6	52.77 7616		1.990 5344	
		51 8171		3 4893
7	53.29 5787		1.987 0451	
		52 0145		3 5057
8	53.81 5932		1.983 5394	
		52 2098		3 5217
9	54.33 8030		1.980 0177	
		52 4029		3 5376
0.510	54.86 2059		1.976 4801	
		52 5942		−3 5529
1	55.38 8001		1.972 9272	
		52 7831		3 5681
2	55.91 5832		1.969 3591	
		52 9702		3 5829
3	56.44 5534		1.965 7762	
		53 1549		3 5974
4	56.97 7083		1.962 1788	
		53 3376		3 6116
0.515	57.51 0459		1.958 5672	
		53 5182		−3 6255
6	58.04 5641		1.954 9417	
		53 6966		3 6391
7	58.58 2607		1.951 3026	
		53 8729		3 6524
8	59.12 1336		1.947 6502	
		54 0471		3 6655
9	59.66 1807		1.943 9847	
		54 2189		3 6782
0.520	60.20 3996		1.940 3065	
		54 3887		−3 6906
1	60.74 7883		1.936 6159	
		54 5564		3 7028
2	61.29 3447		1.932 9131	
		54 7217		3 7147
3	61.84 0664		1.929 1984	
		54 8849		3 7264
4	62.38 9513		1.925 4720	
		55 0458		3 7376
0.525	62.93 9971		1.921 7344	
		55 2047		−3 7487
6	63.49 2018		1.917 9857	
		55 3612		3 7596
7	64.04 5630		1.914 2261	
		55 5156		3 7700
8	64.60 0786		1.910 4561	
		55 6677		3 7804
9	65.15 7463		1.906 6757	
		55 8175		3 7903

TABLE VII—*Continued*

λ (micron)	A' ($T = 2842°K$)		B' ($T = 7000°K$)	
	$J(\lambda)$ (watt m^{-2} micron^{-1})	$\Delta J(\lambda)$	$J(\lambda)$ (watt m^{-2} micron^{-1})	$\Delta J(\lambda)$
0.530	65.71 5638		1.902 8854	
		55 9652		−3 8002
1	66.27 5290		1.899 0852	
		56 1106		3 8096
2	66.83 6396		1.895 2756	
		56 2537		3 8189
3	67.39 8933		1.891 4567	
		56 3946		3 8279
4	67.96 2879		1.887 6288	
		56 5333		3 8366
0.535	68.52 8212		1.883 7922	
		56 6696		−3 8452
6	69.09 4908		1.879 9470	
		56 8037		3 8535
7	69.66 2945		1.876 0935	
		56 9356		3 8615
8	70.23 2301		1.872 2320	
		57 0652		3 8693
9	70.80 2953		1.868 3627	
		57 1925		3 8768
0.540	71.37 4878		1.864 4859	
		57 3175		−3 8843
1	71.94 8053		1.860 6016	
		57 4403		3 8913
2	72.52 2456		1.856 7103	
		57 5607		3 8983
3	73.09 8063		1.852 8120	
		57 6790		3 9049
4	73.67 4853		1.848 9071	
		57 7949		3 9114
0.545	74.25 2802		1.844 9957	
		57 9085		−3 9176
6	74.83 1887		1.841 0781	
		58 0200		3 9236
7	75.41 2087		1.837 1545	
		58 1290		3 9294
8	75.99 3377		1.833 2251	
		58 2358		3 9351
9	76.57 5735		1.829 2900	
		58 3403		3 9404
0.550	77.15 9138		1.825 3496	
		58 4427		−3 9457
1	77.74 3565		1.821 4039	
		58 5425		3 9506
2	78.32 8990		1.817 4533	
		58 6403		3 9554
3	78.91 5393		1.813 4979	
		58 7356		3 9600
4	79.50 2749		1.809 5379	
		58 8288		3 9643

TABLE VII—*Continued*

λ (micron)	A' ($T = 2842°K$) $J(\lambda)$ (watt m^{-2} micron^{-1})	$\Delta J(\lambda)$	B' ($T = 7000°K$) $J(\lambda)$ (watt m^{-2} micron^{-1})	$\Delta J(\lambda)$
0.555	80.09 1037		1.805 5736	
		58 9197		−3 9686
6	80.68 0234		1.801 6050	
		59 0083		3 9726
7	81.27 0317		1.797 6324	
		59 0945		3 9764
8	81.86 1262		1.793 6560	
		59 1787		3 9801
9	82.45 3049		1.789 6759	
		59 2604		3 9835
0.560	83.04 5653		1.785 6924	
		59 3399		−3 9868
1	83.63 9052		1.781 7056	
		59 4172		3 9898
2	84.23 3224		1.777 7158	
		59 4921		3 9928
3	84.82 8145		1.773 7230	
		59 5650		3 9955
4	85.42 3795		1.769 7275	
		59 6354		3 9981
0.565	86.02 0149		1.765 7294	
		59 7037		−4 0005
6	86.61 7186		1.761 7289	
		59 7696		4 0027
7	87.21 4882		1.757 7262	
		59 8335		4 0047
8	87.81 3217		1.753 7215	
		59 8950		4 0066
9	88.41 2167		1.749 7149	
		59 9542		4 0084
0.570	89.01 1709		1.745 7065	
		60 0114		−4 0099
1	89.61 1823		1.741 6966	
		60 0662		4 0114
2	90.21 2485		1.737 6852	
		60 1188		4 0126
3	90.81 3673		1.733 6726	
		60 1693		4 0137
4	91.41 5366		1.729 6589	
		60 2175		4 0146
0.575	92.01 7541		1.725 6443	
		60 2635		−4 0155
6	92.62 0176		1.721 6288	
		60 3073		4 0161
7	93.22 3249		1.717 6127	
		60 3490		4 0165
8	93.82 6739		1.713 5962	
		60 3884		4 0170
9	94.43 0623		1.709 5792	
		60 4258		4 0171

TABLE VII—*Continued*

λ (micron)	A' ($T = 2842°K$) $J(\lambda)$ (watt m^{-2} micron^{-1})	$\Delta J(\lambda)$	B' ($T = 7000°K$) $J(\lambda)$ (watt m^{-2} micron^{-1})	$\Delta J(\lambda)$
0.580	95.03 4881		1.705 5621	
		60 4608		−4 0172
1	95.63 9489		1.701 5449	
		60 4938		4 0172
2	96.24 4427		1.697 5277	
		60 5245		4 0169
3	96.84 9672		1.693 5108	
		60 5532		4 0166
4	97.45 5204		1.689 4942	
		60 5798		4 0161
0.585	98.06 1002		1.685 4781	
		60 6040		−4 0155
6	98.66 7042		1.681 4626	
		60 6264		4 0148
7	99.27 3306		1.677 4478	
		60 6464		4 0138
8	99.87 9770		1.673 4340	
		60 6644		4 0129
9	100.4 8641		1.669 4211	
		6 0681		4 0118
0.590	101.0 9322		1.665 4093	
		6 0694		−4 0105
1	101.7 0016		1.661 3988	
		6 0706		4 0091
2	102.3 0722		1.657 3897	
		6 0715		4 0077
3	102.9 1437		1.653 3820	
		6 0723		4 0060
4	103.5 2160		1.649 3760	
		6 0729		4 0043
0.595	104.1 2889		1.645 3717	
		6 0732		−4 0025
6	104.7 3621		1.641 3692	
		6 0733		4 0005
7	105.3 4354		1.637 3687	
		6 0733		3 9984
8	105.9 5087		1.633 3703	
		6 0730		3 9963
9	106.5 5817		1.629 3740	
		6 0725		3 9939
0.600	107.1 6542		1.625 3801	
		6 0719		−3 9916
1	107.7 7261		1.621 3885	
		6 0710		3 9891
2	108.3 7971		1.617 3994	
		6 0699		3 9865
3	108.9 8670		1.613 4129	
		6 0687		3 9837
4	109.5 9357		1.609 4292	
		6 0672		3 9809

TABLE VII—*Continued*

λ (micron)	A' ($T = 2842°K$) $J(\lambda)$ (watt m^{-2} micron^{-1})	$\Delta J(\lambda)$	B' ($T = 7000°K$) $J(\lambda)$ (watt m^{-2} micron^{-1})	$\Delta J(\lambda)$
0.605	110.2 0029		1.605 4483	
		6 0655		−3 9780
6	110.8 0684		1.601 4703	
		6 0637		3 9750
7	111.4 1321		1.597 4953	
		6 0616		3 9719
8	112.0 1937		1.593 5234	
		6 0593		3 9687
9	112.6 2530		1.589 5547	
		6 0570		3 9653
0.610	113.2 3100		1.585 5894	
		6 0543		−3 9620
1	113.8 3643		1.581 6274	
		6 0515		3 9585
2	114.4 4158		1.577 6689	
		6 0484		3 9549
3	115.0 4642		1.573 7140	
		6 0453		3 9512
4	115.6 5095		1.569 7628	
		6 0419		3 9475
0.615	116.2 5514		1.565 8153	
		6 0383		−3 9437
6	116.8 5897		1.561 8716	
		6 0346		3 9397
7	117.4 6243		1.557 9319	
		6 0306		3 9356
8	118.0 6549		1.553 9963	
		6 0265		3 9316
9	118.6 6814		1.550 0647	
		5 0221		3 9275
0.620	119.2 7035		1.546 1372	
		6 0177		−3 9231
1	119.8 7212		1.542 2141	
		6 0131		3 9188
2	120.4 7343		1.538 2953	
		6 0082		3 9144
3	121.0 7425		1.534 3809	
		6 0031		3 9099
4	121.6 7456		1.530 4710	
		5 9980		3 9054
0.625	122.2 7436		1.526 5656	
		5 9926		−3 9007
6	122.8 7362		1.522 6649	
		5 9870		3 8960
7	123.4 7232		1.518 7689	
		5 9813		3 8912
8	124.0 7045		1.514 8777	
		5 9754		3 8863
9	124.6 6799		1.510 9914	
		5 9694		3 8814

TABLE VII—*Continued*

λ (micron)	A' (T = 2842°K) $J(\lambda)$ (watt m^{-2} micron^{-1})	A' (T = 2842°K) $\Delta J(\lambda)$	B' (T = 7000°K) $J(\lambda)$ (watt m^{-2} micron^{-1})	B' (T = 7000°K) $\Delta J(\lambda)$
0.630	125.2 6493		1.507 1100	
		5 9631		−3 8764
1	125.8 6124		1.503 2336	
		5 9567		3 8714
2	126.4 5691		1.499 3622	
		5 9501		3 8662
3	127.0 5192		1.495 4960	
		5 9434		3 8610
4	127.6 4626		1.491 6350	
		5 9365		3 8558
0.635	128.2 3991		1.487 7792	
		5 9294		−3 8504
6	128.8 3285		1.483 9288	
		5 9222		3 8451
7	129.4 2507		1.480 0837	
		5 9148		3 8396
8	130.0 1655		1.476 2441	
		5 9073		3 8341
9	130.6 0728		1.472 4100	
		5 8995		3 8285
0.640	131.1 9723		1.468 5815	
		5 8917		−3 8230
1	131.7 8640		1.464 7585	
		5 8836		3 8172
2	132.3 7476		1.460 9413	
		5 8755		3 8115
3	132.9 6231		1.457 1298	
		5 8671		3 8057
4	133.5 4902		1.453 3241	
		5 8586		3 7998
0.645	134.1 3488		1.449 5243	
		5 8500		−3 7940
6	134.7 1988		1.445 7303	
		5 8412		3 7880
7	135.3 0400		1.441 9423	
		5 8322		3 7820
8	135.8 8722		1.438 1603	
		5 8232		3 7760
9	136.4 6954		1.434 3843	
		5 8139		3 7698
0.650	137.0 5093		1.430 6145	
		5 8045		−3 7637
1	137.6 3138		1.426 8508	
		5 7950		3 7575
2	138.2 1088		1.423 0933	
		5 7854		3 7513
3	138.7 8942		1.419 3420	
		5 7755		3 7449
4	139.3 6697		1.415 5971	
		5 7655		3 7386

TABLE VII—*Continued*

λ (micron)	A' ($T = 2842°K$)		B' ($T = 7000°K$)	
	$J(\lambda)$ (watt m^{-2} micron^{-1})	$\Delta J(\lambda)$	$J(\lambda)$ (watt m^{-2} micron^{-1})	$\Delta J(\lambda)$
0.655	139.9 4352		1.411 8585	
		5 7555		−3 7323
6	140.5 1907		1.408 1262	
		5 7452		3 7258
7	141.0 9359		1.404 4004	
		5 7348		3 7194
8	141.6 6707		1.400 6810	
		5 7244		3 7128
9	142.2 3951		1.396 9682	
		5 7136		3 7063
0.660	142.8 1087		1.393 2619	
		5 7029		−3 6997
1	143.3 8116		1.389 5622	
		5 6920		3 6931
2	143.9 5036		1.385 8691	
		5 6810		3 6864
3	144.5 1846		1.382 1827	
		5 6697		3 6797
4	145.0 8543		1.378 5030	
		5 6585		3 6730
0.665	145.6 5128		1.374 8300	
		5 6470		−3 6662
6	146.2 1598		1.371 1638	
		5 6355		3 6593
7	146.7 7953		1.367 5045	
		5 6238		3 6526
8	147.3 4191		1.363 8519	
		5 6119		3 6456
9	147.9 0310		1.360 2063	
		5 6001		3 6387
0.670	148.4 6311		1.356 5676	
		5 5880		−3 6318
1	149.0 2191		1.352 9358	
		5 5758		3 6248
2	149.5 7949		1.349 3110	
		5 5635		3 6178
3	150.1 3584		1.345 6932	
		5 5511		3 6107
4	150.6 9095		1.342 0825	
		5 5385		3 6037
0.675	151.2 4480		1.338 4788	
		5 5259		−3 5966
6	151.7 9739		1.334 8822	
		5 5132		3 5894
7	152.3 4871		1.331 2928	
		5 5003		3 5823
8	152.8 9874		1.327 7105	
		5 4873		3 5751
9	153.4 4747		1.324 1354	
		5 4742		3 5679

TABLE VII—*Continued*

λ (micron)	A' (T = 2842°K)		B' (T = 7000°K)	
	$J(\lambda)$ (watt m^{-2} micron^{-1})	$\Delta J(\lambda)$	$J(\lambda)$ (watt m^{-2} micron^{-1})	$\Delta J(\lambda)$
0.680	153.9 9489		1.320 5675	
		5 4610		−3 5607
1	154.5 4099		1.317 0068	
		5 4476		3 5534
2	155.0 8575		1.313 4534	
		5 4343		3 5461
3	155.6 2918		1.309 9073	
		5 4207		3 5388
4	156.1 7125		1.306 3685	
		5 4070		3 5315
0.685	156.7 1195		1.302 8370	
		5 3934		−3 5241
6	157.2 5129		1.299 3129	
		5 3794		3 5167
7	157.7 8923		1.295 7962	
		5 3656		3 5094
8	158.3 2579		1.292 2868	
		5 3515		3 5019
9	158.8 6094		1.288 7849	
		5 3373		3 4945
0.690	159.3 9467		1.285 2904	
		5 3231		−3 4870
1	159.9 2698		1.281 8034	
		5 3088		3 4795
2	160.4 5786		1.278 3239	
		5 2943		3 4721
3	160.9 8729		1.274 8518	
		5 2798		3 4645
4	161.5 1527		1.271 3873	
		5 2652		3 4570
0.695	162.0 4179		1.267 9303	
		5 2504		−3 4494
6	162.5 6683		1.264 4809	
		5 2356		3 4418
7	163.0 9039		1.261 0391	
		5 2208		3 4343
8	163.6 1247		1.257 6048	
		5 2057		3 4267
9	164.1 3304		1.254 1781	
		5 1907		3 4191
0.700	164.6 5211		1.250 7590	
		5 1755		−3 4114
1	165.1 6966		1.247 3476	
		5 1603		3 4038
2	165.6 8569		1.243 9438	
		5 1449		3 3961
3	166.2 0018		1.240 5477	
		5 1295		3 3885
4	166.7 1313		1.237 1592	
		5 1140		3 3807

TABLE VII—*Continued*

λ (micron)	A' ($T = 2842°K$)		B' ($T = 7000°K$)	
	$J(\lambda)$ (watt m^{-2} micron^{-1})	$\Delta J(\lambda)$	$J(\lambda)$ (watt m^{-2} micron^{-1})	$\Delta J(\lambda)$
0.705	167.2 2453		1.233 7785	
		5 0984		−3 3731
6	167.7 3437		1.230 4054	
		5 0828		3 3654
7	168.2 4265		1.227 0400	
		5 0670		3 3576
8	168.7 4935		1.223 6824	
		5 0512		3 3499
9	169.2 5447		1.220 3325	
		5 0353		3 3422
0.710	169.7 5800		1.216 9903	
		5 0194		−3 3344
1	170.2 5994		1.213 6559	
		5 0033		3 3267
2	170.7 6027		1.210 3292	
		4 9872		3 3189
3	171.2 5899		1.207 0103	
		4 9709		3 3111
4	171.7 5608		1.203 6992	
		4 9548		3 3033
0.715	172.2 5156		1.200 3959	
		4 9384		−3 2956
6	172.7 4540		1.197 1003	
		4 9220		3 2877
7	173.2 3760		1.193 8126	
		4 9055		3 2799
8	173.7 2815		1.190 5327	
		4 8890		3 2721
9	174.2 1705		1.187 2606	
		4 8724		3 2643
0.720	174.7 0429		1.183 9963	
		4 8557		−3 2565
1	175.1 8986		1.180 7398	
		4 8390		3 2486
2	175.6 7376		1.177 4912	
		4 8221		3 2408
3	176.1 5597		1.174 2504	
		4 8054		3 2329
4	176.6 3651		1.171 0175	
		4 7884		3 2251
0.725	177.1 1535		1.167 7924	
		4 7714		−3 2172
6	177.5 9249		1.164 5752	
		4 7544		3 2094
7	178.0 6793		1.161 3658	
		4 7373		3 2015
8	178.5 4166		1.158 1643	
		4 7202		3 1937
9	179.0 1368		1.154 9706	
		4 7029		3 1858

TABLE VII—*Continued*

λ (micron)	A' ($T = 2842°K$)		B' ($T = 7000°K$)	
	$J(\lambda)$ (watt m^{-2} micron^{-1})	$\Delta J(\lambda)$	$J(\lambda)$ (watt m^{-2} micron^{-1})	$\Delta J(\lambda)$
0.730	179.4 8397		1.151 7848	
		4 6857		−3 1779
1	179.9 5254		1.148 6069	
		4 6684		3 1701
2	180.4 1938		1.145 4368	
		4 6510		3 1622
3	180.8 8448		1.142 2746	
		4 6336		3 1543
4	181.3 4784		1.139 1203	
		4 6161		3 1464
0.735	181.8 0945		1.135 9739	
		4 5985		−3 1386
6	182.2 6930		1.132 8353	
		4 5811		3 1307
7	182.7 2741		1.129 7046	
		4 5634		3 1228
8	183.1 8375		1.126 5818	
		4 5457		3 1149
9	183.6 3832		1.123 4669	
		4 5280		3 1071
0.740	184.0 9112		1.120 3598	
		4 5103		−3 0992
1	184.5 4215		1.117 2606	
		4 4924		3 0913
2	184.9 9139		1.114 1693	
		4 4747		3 0834
3	185.4 3886		1.111 0859	
		4 4567		3 0756
4	185.8 8453		1.108 0103	
		4 4388		3 0677
0.745	186.3 2841		1.104 9426	
		4 4208		−3 0598
6	186.7 7049		1.101 8828	
		4 4029		3 0520
7	187.2 1078		1.098 8308	
		4 3847		3 0441
8	187.6 4925		1.095 7867	
		4 3667		3 0363
9	188.0 8592		1.092 7504	
		4 3486		3 0284
0.750	188.5 2078		1.089 7220	
		4 3304		−3 0205
1	188.9 5382		1.086 7015	
		4 3123		3 0127
2	189.3 8505		1.083 6888	
		4 2940		3 0049
3	189.8 1445		1.080 6839	
		4 2757		2 9970
4	190.2 4202		1.077 6869	
		4 2575		2 9892

TABLE VII—*Continued*

λ (micron)	A' ($T = 2842°K$) $J(\lambda)$ (watt m^{-2} micron^{-1})	$\Delta J(\lambda)$	B' ($T = 7000°K$) $J(\lambda)$ (watt m^{-2} micron^{-1})	$\Delta J(\lambda)$
0.755	190.6 6777		1.074 6077	
		4 2391		−2 9813
6	191.0 9168		1.071 7164	
		4 2208		2 9736
7	191.5 1376		1.068 7428	
		4 2024		2 9657
8	191.9 3400		1.065 7771	
		4 1839		2 9578
9	192.3 5239		1.062 8193	
		4 1656		2 9501
0.760	192.7 6895		1.059 8692	
		4 1470		−2 9423
1	193.1 8365		1.056 9269	
		4 1286		2 9345
2	193.5 9651		1.053 9924	
		4 1100		2 9266
3	194.0 0751		1.051 0658	
		4 0915		2 9189
4	194.4 1666		1.048 1469	
		4 0720		2 9111
0.765	194.8 2396		1.045 2358	
		4 0543		−2 9034
6	195.2 2939		1.042 3324	
		4 0357		2 8956
7	195.6 3296		1.039 4368	
		4 0171		2 8878
8	196.0 3467		1.036 5490	
		3 9984		2 8800
9	196.4 3451		1.033 6690	
		3 9798		2 8724
0.770	196.8 3249		1.030 7966	
		3 9610		2 8646
1	197.2 2859		1.027 9320	
		3 9424		2 8568
2	197.6 2283		1.025 0752	
		3 9236		2 8492
3	198.0 1519		1.022 2260	
		3 9048		2 8414
4	198.4 0567		1.019 3846	
		3 8861		2 8337
0.775	198.7 9428		1.016 5509	
		3 8673		−2 8260
6	199.1 8101		1.013 7249	
		3 8485		2 8184
7	199.5 6586		1.010 9065	
		3 8297		2 8106
8	199.9 4883		1.008 0959	
		3 8108		2 8030
9	200.3 2991		1.005 2929	
		3 7921		2 7954

TABLE VII—*Concluded*

λ (micron)	A' ($T = 2842°K$)		B' ($T = 7000°K$)	
	$J(\lambda)$ (watt m^{-2} micron^{-1})	$\Delta J(\lambda)$	$J(\lambda)$ (watt m^{-2} micron^{-1})	$\Delta J(\lambda)$
0.780	200.7 0912		1.002 4975	
		3 7731		−27 8769
1	201.0 8643		0.9997 0983	
		3 7544		27 8006
2	201.4 6187		0.9969 2977	
		3 7354		27 7242
3	201.8 3541		0.9941 5735	
		3 7166		27 6480
4	202.2 0707		0.9913 9255	
		3 6977		27 5720
0.785	202.5 7684		0.9886 3535	
		3 6787		−27 4959
6	202.9 4471		0.9858 8576	
		3 6599		27 4200
7	203.3 1070		0.9831 4376	
		3 6410		27 3443
8	203.6 7480		0.9804 0933	
		3 6220		27 2686
9	204.0 3700		0.9776 8247	
		3 6032		27 1930
0.790	204.3 9732		0.9749 6317	
		3 5842		−27 1176
1	204.7 5574		0.9722 5141	
		3 5652		27 0422
2	205.1 1226		0.9695 4719	
		3 5464		26 9670
3	205.4 6690		0.9668 5049	
		3 5273		26 8918
4	205.8 1963		0.9641 6131	
		3 5085		26 8168
0.795	206.1 7048		0.9614 7963	
		3 4895		−26 7419
6	206.5 1943		0.9588 0544	
		3 4706		26 6672
7	206.8 6649		0.9561 3872	
		3 4516		26 5925
8	207.2 1165		0.9534 7947	
		3 4327		26 5179
9	207.5 5492		0.9508 2768	
		3 4137		26 4435
0.800	207.8 9629		0.9481 8333	

Thus,

$$\frac{2.1882\ 81837}{\lambda} = 3.94996\ 72148$$

$$10^{(2.1882\ 81837)/\lambda} = 8911.8\ 36595$$

$$3.6970 \times 10^8/\lambda^5 = 70843\ 6.0076$$

$$J(\lambda) = \frac{70843\ 6.0076}{8910.8\ 36595} = 79.502\ 74931$$

(watt m^{-2} micron^{-1}).

5. Color-quality Diagram

In many applications of colorimetry, the absolute magnitude is unimportant and interest centers in the *color quality*. This characteristic is specified by the homogeneous coordinates x, y, and z:

$$x = \frac{X}{X+Y+Z},$$
$$y = \frac{Y}{X+Y+Z}, \tag{21}$$
$$z = \frac{Z}{X+Y+Z}.$$

The spectral locus in the color 2-space is computed by substituting the trichromatic coordinates of homogeneous radiation ($X = \bar{x}$, $Y = \bar{y}$, $Z = \bar{z}$) in Eq. (21). The results are given in Table VIII. Only x and y are listed, since z is dependent on the other two homogeneous coordinates:

$$x + y + z = 1. \tag{22}$$

A plot of x vs y gives the spectral locus of Fig. 5. Also given is the spectral locus plotted from the present C.I.E. data. The saturated purples are represented by the straight line joining the extreme ends of the spectrum:

$$\lambda \to 0 \quad \begin{cases} x = 0.1795\ \ 3321, \\ y = 0. \end{cases}$$

$$\lambda \to \infty \quad \begin{cases} x = 0.7316\ \ 4591, \\ y = 0.2683\ \ 5409. \end{cases}$$

The Planckian locus is also shown in Fig. 5. It was obtained from Eq. (21) by substitution of values from Table V. Data for the Planckian locus are listed in Table IX.

Example.

Determine the homogeneous coordinates for homogeneous radiation at $\lambda = 0.554\mu$.

From Table II,

$$\bar{x} = 0.50306\ 52161$$
$$\bar{y} = 1.03953\ 87260$$
$$\bar{z} = 0.00141\ 09972$$
$$\Sigma = 1.54401\ 49393$$

TABLE VIII

The spectral locus

λ (micron)	x	Δx	y	Δy
0	^{0}1795 3321		0	
		−23	^{1}3444 4372	+53 7154
0.360	^{0}1795 2638			
		26	3498 1526	59 2404
1	1795 2615			
		31	3557 3930	64 9495
2	1795 2589			
		34	3622 3425	70 8840
3	1795 2558			
		40	3693 2265	77 0621
4	1795 2524			
		−46	^{1}3770 2886	83 5219
0.365	^{0}1795 2484			
		52	3853 8105	90 2726
6	1795 2438			
		60	3944 0831	97 3562
7	1795 2386			
		60	4041 4393	104 8005
8	1795 2326			
		78	4146 2398	112 6394
9	1795 2257			
		−90	^{1}4258 8792	120 9061
0.370	^{0}1795 2179			
		102	4379 7853	129 6418
1	1795 2089			
		117	4509 4271	138 8847
2	1795 1987			
		133	4648 3118	148 6751
3	1795 1870			
		152	4796 9869	159 0758
4	1795 1737			
		−173	^{1}4956 0627	170 1130
0.375	^{0}1795 1585			
		198	5126 1757	181 8574
6	1795 1412			
		225	5308 0331	194 3635
7	1795 1214			
		256	5502 3966	207 6945
8	1795 0989			
		290	5710 0911	221 9192
9	1795 0733			

TABLE VIII—*Continued*

λ (micron)	x	Δx	y	Δy
0.380	1795 0443		5932 0103	
		−331		237 1111
1	1795 0112		6169 1214	
		374		253 3525
2	1794 9738		6422 4739	
		425		270 7277
3	1794 9313		6693 2016	
		480		289 3330
4	1794 8833		6982 5346	
		544		309 2707
0.385	1794 8289		7291 8053	
		−614		330 6521
6	1794 7675		7622 4574	
		693		353 5982
7	1794 6982		7976 0556	
		781		378 2411
8	1794 6201		8354 2967	
		879		404 7239
9	1794 5322		8759 0206	
		988		433 2028
0.390	1794 4334		9192 2234	
		−1109		463 8479
1	1794 3225		9656 0713	
		1245		496 8458
2	1794 1980		1015 2916	
		1394		53 2394
3	1794 0586		1068 5310	
		1559		57 0719
4	1793 9027		1125 6029	
		1741		61 2058
0.395	1793 7286		1186 8087	
		−1944		65 6674
6	1793 5342		1252 4761	
		2165		70 4855
7	1793 3177		1322 9616	
		2409		75 6915
8	1793 0768		1398 6531	
		2678		81 3197
9	1792 8090		1479 9728	
		2972		87 4074
0.400	1792 5118		1567 3802	
		−3295		93 9960
1	1792 1823		1661 3762	
		3647		101 1305
2	1791 8176		1762 5067	
		4031		108 8508
3	1791 4145		1871 3665	
		4452		117 2383
4	1790 9693		1988 6048	
		4909		126 3250

TABLE VIII—*Continued*

λ (micron)	x	Δx	y	Δy
0.405	1790 4784		2114 9298	
		−5406		136 1848
6	1789 9378		2251 1146	
		5945		146 8889
7	1789 3433		2398 0035	
		6531		158 5156
8	1788 6902		2556 5191	
		7165		171 1508
9	1787 9737		2727 6699	
		7850		184 8886
0.410	1787 1887		2912 5585	
		−8501		199 8329
1	1786 3296		3112 3914	
		9388		216 0975
2	1785 3908		3328 4889	
		1 0248		233 8078
3	1784 3660		3562 2967	
		1 1173		253 1018
4	1783 2487		3815 3985	
		1 2166		274 1311
0.415	1782 0321		4089 5296	
		−1 3231		297 0633
6	1780 7090		4386 5929	
		1 4373		322 0821
7	1779 2717		4708 6750	
		1 5594		349 3910
8	1777 7123		5058 0660	
		1 6898		379 2133
9	1776 0225		5437 2793	
		1 8292		411 7962
0.420	1774 1933		5849 0755	
		−1 9776		447 4117
1	1772 2157		6296 4872	
		2 1356		486 3608
2	1770 0801		6782 8480	
		2 3036		528 9749
3	1767 7765		7311 8229	
		2 4821		575 6208
4	1765 2944		7887 4437	
		2 6714		626 7035
0.425	1762 6230		8514 1472	
		−2 8719		682 6705
6	1759 7511		9196 8177	
		3 0842		744 0170
7	1756 6669		9940 8347	
		3 3085		811 2902
8	1753 3584		10075 2125	
		3 5454		88 5096
9	1749 8130		1163 7221	
		3 7953		96 6105

TABLE VIII—*Continued*

λ (micron)	x	Δx	y	Δy
0.430	^{9}1746 0177		^{3}1260 3326	
		−4 0584		105 5058
1	1741 9593		1365 8384	
		4 3354		115 2778
2	1737 6239		1481 1162	
		4 6266		126 0175
3	1732 9973		1607 1337	
		4 9324		137 8259
4	1728 0649		1744 9596	
		5 2532		150 8144
0.435	^{9}1722 8117		^{3}1895 7740	
		−5 5894		165 1073
6	1717 2223		2060 8813	
		5 9414		180 8417
7	1711 2809		2241 7230	
		6 3096		198 1700
8	1704 9713		2439 8930	
		6 6942		217 2614
9	1698 2771		2657 1544	
		7 0960		238 3028
0.440	^{9}1691 1811		^{3}2895 4572	
		−7 5148		261 5027
1	1683 6663		3156 9599	
		7 9514		287 0912
2	1675 7149		3444 0511	
		8 4060		315 3247
3	1667 3089		3759 3758	
		8 8788		346 4866
4	1658 4301		4105 8624	
		9 3704		380 8923
0.445	^{9}1649 0597		^{3}4486 7547	
		−9 8808		418 8912
6	1639 1789		4905 6459	
		10 4106		460 8706
7	1628 7683		5366 5165	
		10 9598		507 2604
8	1617 8085		5873 7769	
		11 5291		558 5368
9	1606 2794		6432 3137	
		12 1184		615 2279
0.450	^{9}1594 1610		^{3}7047 5416	
		−12 7282		677 9182
1	1581 4328		7725 4598	
		13 3585		747 2552
2	1568 0743		8472 7150	
		14 0098		823 9549
3	1554 0645		9296 6699	
		14 6822		908 8088
4	1539 3823		^{1}1020 5479	
		15 3759		100 2691

TABLE VIII—*Continued*

λ (micron)	x	Δx	y	Δy
0.455	^{9}1524 0064		^{1}1120 8170	
		−16 0909		110 6567
6	1507 9155		1231 4737	
		16 8277		122 1498
7	1491 0878		1353 6235	
		17 5859		134 8654
8	1473 5019		1488 4889	
		18 3659		148 9319
9	1455 1360		1637 4208	
		19 1675		164 4900
0.460	^{9}1435 9685		^{1}1801 9108	
		−19 9904		181 6937
1	1415 9781		1983 6045	
		20 8345		200 7110
2	1395 1436		2184 3155	
		21 6094		221 7243
3	1373 4442		2406 0398	
		22 5846		244 9316
4	1350 8596		2650 9714	
		23 4890		270 5465
0.465	^{9}1327 3706		^{1}2921 5179	
		−24 4119		298 7980
6	1302 9587		3220 3159	
		25 3519		329 9314
7	1277 6068		3550 2473	
		26 3073		364 2062
8	1251 2995		3914 4535	
		27 2758		401 8958
9	1224 0237		4316 3493	
		28 2550		443 2854
0.470	^{9}1195 7687		^{1}4759 6347	
		−29 2416		488 6683
1	1166 5271		5248 3030	
		30 2316		538 3428
2	1136 2955		5786 6458	
		31 2203		592 6063
3	1105 0752		6379 2521	
		32 2020		651 7484
4	1072 8732		7031 0005	
		33 1700		716 0417
0.475	^{9}1039 7032		^{1}7747 0422	
		−34 1164		785 7320
6	1005 5868		8532 7742	
		350 3183		861 0240
7	^{1}9705 5497		9393 7982	
		359 0580		942 0670
8	9346 4917		^{9}1033 5865	
		367 2604		102 8937
9	8979 2313		1136 4802	
		374 7875		112 1613

TABLE VIII—*Continued*

λ (micron)	x	Δx	y	Δy
0.480	^{1}8604 4438		^{0}1248 6415	
		−381 4854		121 9964
1	8222 9584		1370 6379	
		387 1836		132 3717
2	7835 7748		1503 0096	
		391 6970		143 2434
3	7444 0778		1646 2530	
		394 8271		154 5494
4	7049 2507		1800 8024	
		396 3656		166 2070
0.485	^{1}6652 8851		^{0}1967 0094	
		−396 0979		178 1111
6	6256 7872		2145 1205	
		393 8098		190 1335
7	5862 9774		2335 2540	
		389 2925		202 1226
8	5473 6849		2537 3766	
		382 3529		213 9049
9	5091 3320		2751 2815	
		372 8201		225 2871
0.490	^{1}4718 5119		^{0}2976 5686	
		−360 5569		236 0606
1	4357 9550		3212 6292	
		345 4680		246 0064
2	4012 4870		3458 6356	
		327 5088		254 9029
3	3684 9782		3713 5385	
		306 6935		262 5336
4	3378 2847		3976 0721	
		283 0988		268 6962
0.495	^{1}3095 1859		^{0}4244 7683	
		−256 8670		273 2126
6	2838 3189		4517 9809	
		228 2039		275 9368
7	2610 1150		4793 9177	
		197 3745		276 7635
8	2412 7405		5070 6812	
		164 6936		275 6333
9	2248 0469		5346 3145	
		130 5144		272 5373
0.500	^{1}2117 5325		^{0}5618 8518	
		−95 2147		267 5165
1	2022 3178		5886 3683	
		−59 1808		260 6615
2	1963 1370		6147 0298	
		−22 7925		252 1064
3	1940 3445		6399 1362	
		+13 5914		242 0231
4	1953 9359		6641 1593	
		+49 6464		230 6120

TABLE VIII—*Continued*

λ (micron)	x	Δx	y	Δy
0.505	12003 5823		16871 7713	
		+85 0915		218 0930
6	2088 6738		7089 8643	
		119 6959		204 6956
7	2208 3697		7294 5599	
		153 2825		190 6500
8	2361 6522		7485 2099	
		185 7281		176 1786
9	2547 3803		7661 3885	
		216 9602		161 4896
0.510	12764 3405		17822 8781	
		246 9529		146 7715
1	3011 2934		7969 6496	
		275 7198		132 1904
2	3287 0132		8101 8400	
		303 3075		117 8870
3	3590 3207		8219 7270	
		329 7877		103 9770
4	3920 1084		8323 7040	
		355 2491		90 5520
0.515	14275 3575		18414 2560	
		379 7917		77 6798
6	4655 1492		8491 9358	
		403 5198		65 4082
7	5058 6690		8557 3440	
		426 5372		53 7661
8	5485 2062		8611 1101	
		448 9426		42 7673
9	5934 1488		8653 8774	
		470 8273		32 4126
0.520	16404 9761		18686 2900	
		492 2713		22 6923
1	6897 2474		8708 9823	
		513 3431		13 5893
2	7410 5905		8722 5716	
		534 0979		+5 0799
3	7944 6884		8727 6515	
		554 5781		−2 8630
4	8499 2665		8724 7885	
		574 8122		10 2693
0.525	19074 0787		18714 5192	
		594 8166		−17 1699
6	9668 8953		8697 3493	
		614 5957		23 5963
7	21028 3491		8673 7530	
		63 4142		29 5791
8	1091 7633		8644 1739	
		65 3439		35 1480
9	1157 1072		8609 0259	
		67 2462		40 3310

TABLE VIII—*Continued*

λ (micron)	x	Δx	y	Δy
0.530	^{0}1224 3534		^{0}8568 6949	
		69 1176		−45 1544
1	1293 4710		8523 5405	
		70 9544		49 6424
2	1364 4254		8473 8981	
		72 7518		53 8172
3	1437 1772		8420 0809	
		74 5051		57 6992
4	1511 6823		8362 3817	
		76 2091		61 3062
0.535	^{0}1587 8914		^{0}8301 0755	
		77 8583		−64 6548
6	1665 7497		8236 4207	
		79 4475		67 7592
7	1745 1972		8168 6615	
		80 9710		70 6325
8	1826 1682		8098 0290	
		82 4235		73 2861
9	1908 5917		8024 7429	
		83 7998		75 7300
0.540	^{0}1992 3915		^{0}7949 0129	
		85 0952		−77 9733
1	2077 4867		7871 0396	
		86 3047		80 0239
2	2163 7914		7791 0157	
		87 4243		81 8888
3	2251 2157		7709 1269	
		88 4500		83 5749
4	2339 6657		7625 5520	
		89 3786		85 0875
0.545	^{0}2429 0443		^{0}7540 4645	
		90 2070		−86 4324
6	2519 2513		7454 0321	
		90 9328		87 6146
7	2610 1841		7366 4175	
		91 5544		88 6395
8	2701 7385		7277 7780	
		92 0697		89 5093
9	2793 8082		7188 2687	
		92 4790		90 2323
0.550	^{0}2886 2872		^{0}7098 0364	
		92 7811		−90 8105
1	2979 0683		7007 2259	
		92 9768		91 2491
2	3072 0451		6915 9768	
		93 0664		91 5526
3	3165 1115		6824 4242	
		93 0514		91 7256
4	3258 1629		6732 6986	
		92 9334		91 7728

TABLE VIII—*Continued*

λ (micron)	x	Δx	y	Δy
0.555	.3351 0963		.6640 9258	
		92 7145		−91 6989
6	3443 8108		6549 2269	
		92 3973		91 5091
7	3536 2081		6457 7178	
		91 9845		91 2080
8	3628 1926		6366 5098	
		91 4793		90 8007
9	3719 6719		6275 7091	
		90 8853		90 2926
0.560	.3810 5572		.6185 4165	
		90 2062		−89 6886
1	3900 7634		6095 7279	
		89 4459		88 9941
2	3990 2093		6006 7338	
		88 6084		88 2142
3	4078 8177		5918 5196	
		87 6982		87 3542
4	4166 5159		5831 1654	
		86 7194		86 4194
0.565	.4253 2353		.5744 7458	
		85 6766		−85 4151
6	4338 9119		5659 3307	
		84 5741		84 3463
7	4423 4860		5574 9844	
		83 4165		83 2180
8	4506 9025		5491 7664	
		82 2082		82 0353
9	4589 1107		5409 7311	
		80 9535		80 8029
0.570	.4670 0642		.5328 9282	
		79 6569		−79 5258
1	4749 7211		5249 4024	
		78 3224		78 2083
2	4828 0435		5171 1941	
		76 9542		76 8548
3	4904 9977		5094 3393	
		75 5563		75 4700
4	4980 5540		5018 8693	
		74 1327		74 0575
0.575	.5054 6867		.4944 8118	
		72 6869		−72 6215
6	5127 3736		4872 1903	
		71 2225		71 1656
7	5198 5961		4801 0247	
		69 7429		69 6933
8	5268 3390		4731 3314	
		68 2511		68 2082
9	5336 5901		4663 1232	
		66 7505		66 7131

TABLE VIII—*Continued*

λ (micron)	x	Δx	y	Δy
0.580	5403 3406		4596 4101	
		65 2437		−65 2112
1	5468 5843		4531 1989	
		63 7334		63 7051
2	5532 3177		4467 4938	
		62 2221		62 1975
3	5594 5398		4405 2963	
		60 7121		60 6907
4	5655 2519		4344 6056	
		59 2055		59 1870
0.585	5714 4574		4285 4186	
		57 7045		−57 6883
6	5772 1619		4227 7303	
		56 2107		56 1966
7	5828 3726		4171 5337	
		54 7257		54 7135
8	5883 0983		4116 8202	
		53 2512		53 2406
9	5936 3495		4063 5796	
		51 7884		51 7792
0.590	5988 1379		4011 8004	
		50 3386		−50 3306
1	6038 4765		3961 4698	
		48 9028		48 8959
2	6087 3793		3912 5739	
		47 4821		47 4760
3	6134 8614		3865 0979	
		46 0772		46 0718
4	6180 9386		3819 0261	
		44 6887		44 6842
0.595	6225 6273		3774 3419	
		43 3175		−43 3135
6	6268 9448		3731 0284	
		41 9639		41 9604
7	6310 9087		3689 0680	
		40 6284		40 6254
8	6351 5371		3648 4426	
		39 3112		39 3086
9	6390 8483		3609 1340	
		38 0127		38 0104
0.600	6428 8610		3571 1236	
		36 7331		−36 7311
1	6465 5941		3534 3925	
		35 4723		35 4706
2	6501 0664		3498 9219	
		34 2305		34 2290
3	6535 2969		3464 6929	
		33 0076		33 0063
4	6568 3045		3431 6866	
		31 8035		31 8024

TABLE VIII—*Continued*

λ (micron)	x	Δx	y	Δy
0.605	*0600 1080		*3399 8842	
		30 6182		−30 6172
6	6630 7262		3369 2670	
		29 4514		29 4505
7	6660 1776		3339 8165	
		28 3029		28 3022
8	6688 4805		3311 5143	
		27 1726		27 1718
9	6715 6531		3284 3425	
		26 0599		26 0594
0.610	*6741 7130		*3258 2831	
		24 9648		−24 9643
1	6766 6778		3233 3188	
		23 8868		23 8864
2	6790 5646		3209 4324	
		22 8157		22 8253
3	6813 3903		3186 6071	
		21 7908		21 7805
4	6835 1711		3164 8266	
		20 7520		20 7517
0.615	*6855 9231		*3144 0749	
		19 7387		−19 7384
6	6875 6618		3124 3365	
		18 7406		18 7405
7	6894 4024		3105 5960	
		17 7573		17 7570
8	6912 1597		3087 8390	
		16 7880		16 7879
9	6928 9477		3071 0511	
		15 8327		15 8326
0.620	*6944 7804		*3055 2185	
		14 8907		−14 8695
1	6959 6711		3040 3280	
		13 9615		13 9823
2	6973 6326		3026 3667	
		13 0446		13 0446
3	6986 6772		3013 3221	
		12 1398		12 1397
4	6998 8170		3001 1824	
		11 2463		11 2463
0.625	*7010 0633		*2989 9361	
		10 3644		−10 3643
6	7020 4277		2979 5718	
		9 4986		9 4985
7	7029 9263		2970 0733	
		8 6712		8 6712
8	7038 5975		2961 4021	
		7 9336		7 9335
9	7046 5311		2953 4686	
		7 3574		7 3574

TABLE VIII—*Continued*

λ (micron)	x	Δx	y	Δy
0.630	7053 8885		2946 1112	
		7 0071		−7 0071
1	7060 8956		2939 1041	
		6 9145		6 9144
2	7067 8101		2932 1897	
		7 0693		7 0693
3	7074 8794		2925 1204	
		7 4271		7 4271
4	7082 3065		2917 6933	
		7 9236		7 9236
0.635	7090 2301		2909 7697	
		8 4902		−8 4901
6	7098 7203		2901 2796	
		9 0638		9 0638
7	7107 7841		2892 2158	
		9 5943		9 5943
8	7117 3784		2882 6215	
		10 0446		10 0446
9	7127 4230		2872 5769	
		10 3918		10 3918
0.640	7137 8148		2862 1851	
		10 6242		−10 6242
1	7148 4390		2851 5609	
		10 7389		10 7389
2	7159 1779		2840 8220	
		10 7399		10 7398
3	7169 9178		2830 0822	
		10 6353		10 6353
4	7180 5531		2819 4469	
		10 4363		10 4363
0.645	7190 9894		2809 0106	
		10 1552		−10 1552
6	7201 1446		2798 8554	
		9 8049		9 8049
7	7210 9495		2789 0505	
		9 3980		9 3980
8	7220 3475		2779 6525	
		8 9467		8 9467
9	7229 2942		2770 7057	
		8 4621		8 4620
0.650	7237 7563		2762 2437	
		7 9543		−7 9543
1	7245 7106		2754 2894	
		7 4325		7 4325
2	7253 1431		2746 8569	
		6 9050		6 9050
3	7260 0481		2739 9519	
		6 3787		6 3787
4	7266 4268		2733 5732	
		5 8603		5 8603

TABLE VIII—*Concluded*

λ (micron)	x	Δx	y	Δy
0.655	·7272 2871		·2727 7129	
		5 3551		−5 3552
6	7277 6422		2722 3578	
		4 8684		4 8684
7	7282 5106		2717 4894	
		4 4044		4 4044
8	7286 9150		2713 0850	
		3 9668		3 9668
9	7290 8818		2709 1182	
		3 5597		3 5597
0.660	·7294 4415		·2705 5585	
		3 1859		−3 1859
1	7297 6274		2702 3726	
		2 8487		2 8487
2	7300 4761		2699 5239	
		2 5510		2 5510
3	7303 0271		2696 9729	
		2 2954		2 2954
4	7305 3225		2694 6775	
		2 0851		2 0851
0.665	·7307 4076		·2692 5924	
		1 9226		−1 9226
6	7309 3302		2690 6698	
		1 8109		1 8109
7	7311 1411		2688 8589	
		1 7531		1 7531
8	7312 8942		2687 1058	
		1 7525		1 7525
9	7314 6467		2685 3533	
		1 8124		1 8124
0.670	·7316 4591		·2683 5409	

Thus

$$x = \frac{0.50306\ 52161}{1.54401\ 49393} = 0.32581\ 62880.$$

Similarly,

$$y = 0.67326\ 98627,$$

$$z = 0.00091\ 38493.$$

The limiting point ($T \rightarrow \infty$) on the Planckian locus is obtained from the equation,[5]

$$L = \frac{AC_1T}{C_2} \frac{\Gamma(p+3)}{q^{p+3}}.$$

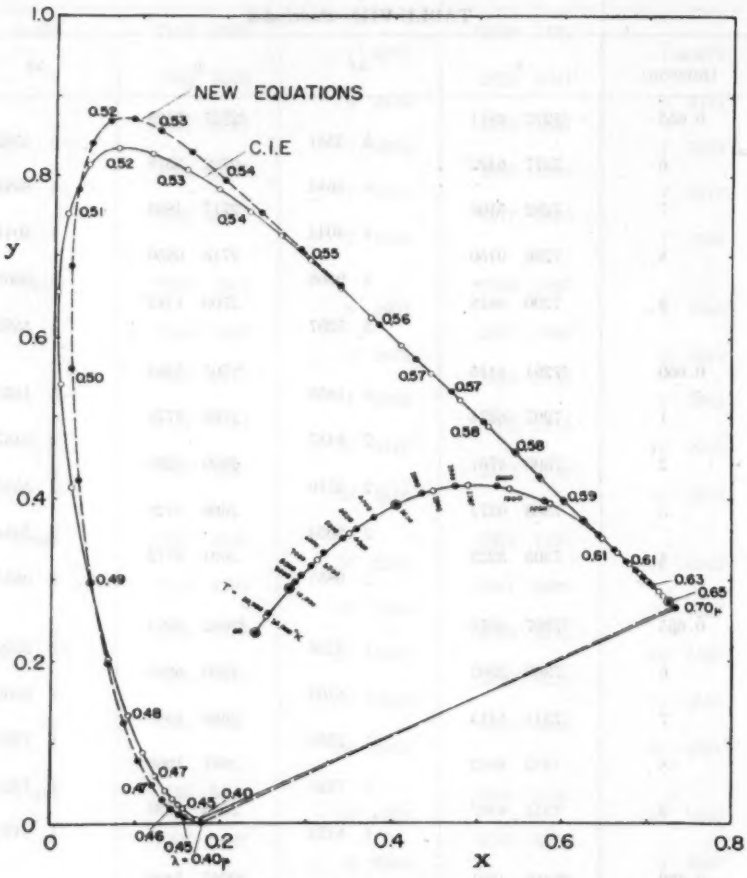

FIG. 5. Color diagram,
○ Standard C.I.E. values
● Analytic approximation.

When equations of this form are substituted in Eq. (21), the T cancels and very simple expressions are obtained:

$$z = \frac{\dfrac{A_s \Gamma(p_s + 3)}{q^{p_s+3}}}{\sum \dfrac{A\Gamma(p+3)}{q^{p+3}}} \tag{23}$$

where the summation is for the five sets of values corresponding to $\bar{z}$, $\bar{x}_A$, $\bar{x}_B$, $\bar{x}_C$, $\bar{y}_A$, $\bar{y}_D$. A similar expression is written for x and y.

TABLE IX

The Planckian locus

T (°K)	$1/T$ (mireds)	x	y
1000	1000	6514 4365	3461 2262
1023	977	6485 4731	3486 3625
1125	888	6353 7467	3596 1156
1143	875	6330 3568	3614 7577
1200	833	6255 2378	3672 7582
1286	777	6142 4396	3754 1709
1333	750	6080 0040	3796 1351
1400	714	5974 9893	3834 0429
1421	704	5965 9344	3866 8305
1500	667	5864 4502	3923 0450
1556	643	5793 9886	3958 3034
1600	625	5738 2128	3984 0090
1750	571	5554 3733	4054 9295
1800	555	5494 7157	4073 4352
2000	500	5265 2752	4124 3322
2046	488	5214 6680	4131 3289
2222	450	5028 6115	4144 4647
2250	445	5000 3223	4144 7243
2333	429	4917 9424	4142 7714
2500	400	4761 3765	4129 1564
2667	375	4616 0805	4104 8264
2800	357	4507 6797	4079 5339
2842	352	4474 9332	4070 7118
2857	350	4463 2962	4067 4590
3000	333	4357 5018	4034 6041
3333	300	4137 5732	3948 4000
3500	286	4040 4111	3902 6666
4000	250	3792 1655	3764 5414
4500	222	3596 5597	3633 8633
4667	214	3540 7445	3592 9598
5000	200	3440 7836	3515 5938
6000	167	3213 4917	3318 2047
6667	150	3103 1582	3213 2026
7000	143	3057 4332	3167 0569
8000	125	2947 0149	3050 0625
9000	111	2865 5387	2958 5285
10,000	100	2803 5364	2885 7010
14,000	71.4	2658 8076	2704 4116
20,000	50.0	2565 6398	2578 7790
∞	0	2402 6941	2341 4157

Example.

What are the homogeneous coordinates for Planckian radiation at $T \to \infty$? For $\bar{x}(\lambda)$,

$$\begin{aligned} \log A &= 32.75260\ 000 \\ \log \Gamma(p+3) &= 784.39706\ 934 \\ -(p+3)\log q &= -816.73747\ 455 \end{aligned}$$

$$\log\left[\frac{A\Gamma(p+3)}{q^{p+3}}\right] = 0.41219\ 479$$

$$\text{or} \quad \frac{A\Gamma(p+3)}{q^{p+3}} = 2.5834\ 186$$

Similarly,

$$\text{for } \bar{x}_A, \quad \frac{A\Gamma(p+3)}{q^{p+3}} = 0.56529\ 947$$

$$\begin{aligned} \bar{x}_C, &\quad 0.72525\ 263 \\ \bar{x}_B, &\quad 0.10956\ 003 \\ \bar{y}_A, &\quad 1.15454\ 573 \\ \bar{y}_D, &\quad 0.00367\ 370. \end{aligned}$$

$$\Sigma = 4.91530\ 883.$$

Thus,

$$x = \frac{0.56529\ 947 + 0.72525\ 263 - 0.10956\ 003}{4.91530\ 883}$$

$$= 0.24026\ 813$$

$$y = \frac{1.15087\ 203}{4.91530\ 883} = 0.23414\ 0565$$

6. Power Functions

Many reflectance and transmittance curves can be approximated by a power function $K_m\lambda^m$ or by a few terms of a power series, For example, most woods have spectral reflectance curves[6] that are almost straight lines, as shown in Fig. 6. Most white surfaces are represented by an equation[6] of the form,

$$\rho(\lambda) = A - B\lambda^m.$$

An example is shown in Fig. 7. Similar results can be obtained for the spectral transmittance $\tau(\lambda)$ of transparent and translucent materials.

According to Table III, the colorimetric integrations for Wien radiation reflected from a surface whose reflectance is expressed by a power function are of the form:

$$L/K_m = AC_1\frac{\Gamma(p+4-m)}{(q+nC_2/T)^{p+4-m}}. \tag{24}$$

For a given function ($\bar{z}$, $\bar{x}_B$, $\bar{x}_C$, $\bar{x}_A$, $\bar{y}_A$, or $\bar{y}_D$), *all the quantities on the right side of Eq.* (24) *are known constants.* Thus the quantity (Z/K_m) can be tabulated

as a function of m, and these tabulated values can be used in obtaining Z for any value of K_m:

$$Z = (Z/K_m)K_m . \tag{25}$$

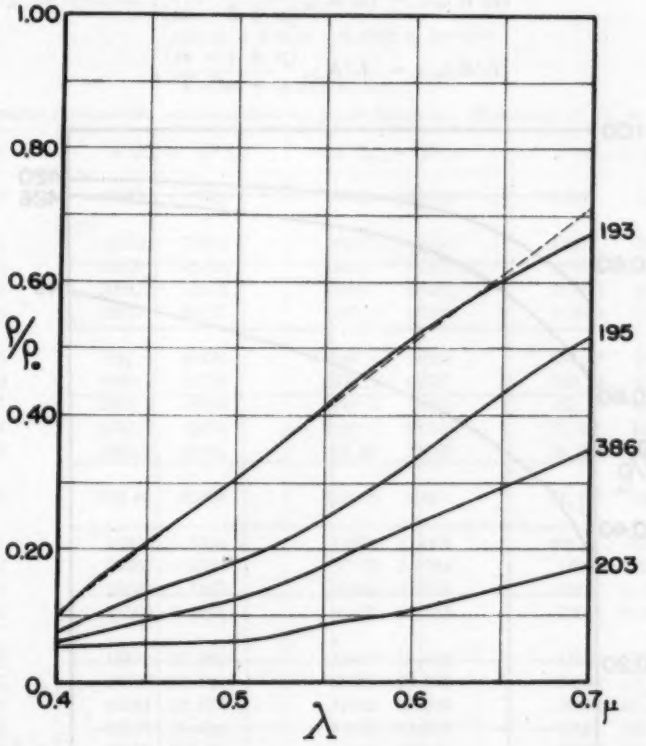

FIG. 6. Reflectance curves for woods.
Sample No. 193—Prima vera, varnished but unstained. Solid curve experimental, dotted curve plotted from the equation,

$$\rho/\rho_0 = -0.717 + 2.043\,\lambda.$$

195—White oak, varnished but unstained.
386—White oak, stained.
203—Oriental wood, varnished but unstained.

The same procedure applies to the determination of (X/K_m) and (Y/K_m), except for the slight additional complication:

$$Y/K_m = Y_A/K_m - Y_D/K_m ,$$

$$X/K_m = X_A/K_m + X_C/K_m - X_B/K_m . \tag{26}$$

If instead of Wien radiation one has Planckian radiation, the same method applies but more terms must be calculated in accordance with the formulas of

Table III. Calculated values of X/K_m, Y/K_m, and Z/K_m are listed in Table X for m from -10 to $+10$. The recurrence formulas,

$$(L/K)_{m+1} = (L/K)_m \frac{q + nC_2/T}{(p + 3 - m)},$$
$$(L/K)_{m-1} = (L/K)_m \frac{(p + 4 - m)}{q + nC_2/T}, \tag{27}$$

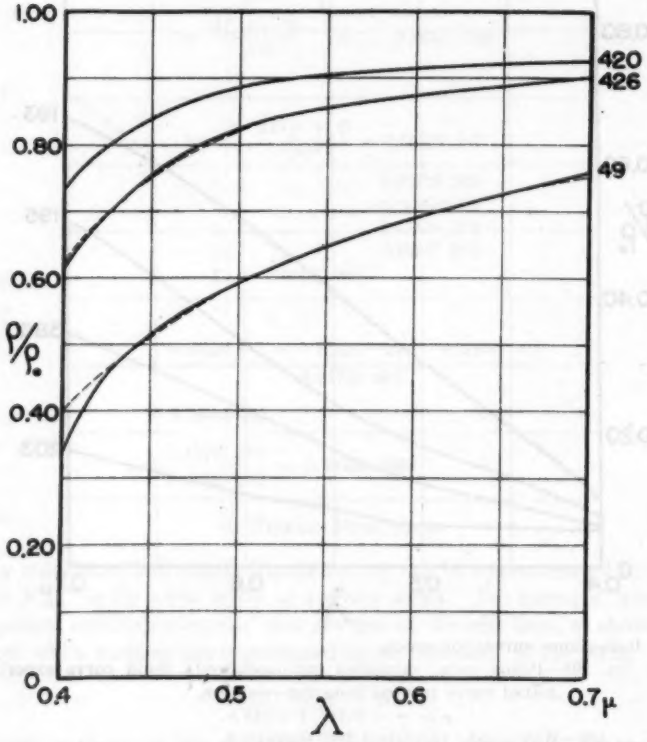

FIG. 7. Reflectance curves for white materials. Solid curves are experimental, dotted curves are approximations:

No. 420—Matt white paint,
$$\rho/\rho_0 = 0.9265 - 3.2195 \times 10^{-4}/\lambda^3.$$
426—Semi-luster white paint,
$$\rho/\rho_0 = 0.9137 - 2.9842 \times 10^{-3}/\lambda^3.$$
49—Acoustical tile,
$$\rho/\rho_0 = 0.9229 - 8.3612 \times 10^{-3}/\lambda^2.$$

were employed, checked whenever necessary by direct calculations with the integrals of Table III. The results apply to incident Planckian radiation at

2842° and 7000°K. Calculations can be made if needed, however, for other illuminants.

For example, consider a white paint (No. 419, Reference 10). The reflectance is represented by the equation,

$$\rho(\lambda)/\rho_0 = 0.9116 - 8.3488 \times 10^{-6}/\lambda^{10}.$$

TABLE X-A

Coefficients for analytic representation by power functions, illuminant A' (T = 2842°K)

m	$X/K_m = (G^m)_x$	$Y/K_m = (G^m)_y$	$Z/K_m = (G^m)_z$
0	97871. 15758	89030. 43140	31808. 15913
1	58213. 74203	50837. 46164	14679. 55636
2	34829. 43400	29177. 27551	6793.0 99875
3	20947. 43173	16831. 41425	3152.1 50622
4	12657. 68727	9759.1 27328	1466.6 72173
5	7681.4 56236	5687.4 24315	684.30 47043
6	4680.2 08759	3331.4 62027	320.15 45319
7	2862.3 19093	1961.4 16866	150.19 88913
8	1756.8 13078	1160.7 08641	70.660 06836
9	1082.0 08061	690.39 71133	33.333 81029
10	668.63 69908	412.76 55677	15.768 95051
−1	16565 1.6324	15671 4.4008	69110. 11449
−2	28255 1.4794	27726 5.0264	15056 3.2492
−3	48630 2.7495	49305 6.2803	32890 2.7208
−4	84580 6.4444	88127 6.0923	72041 6.9499
−5	14891 32.965	15832 14.636	15822 13.994
−6	26589 54.108	28587 58.385	34842 41.456
−7	48244 93.973	51882 60.618	76932 50.787
−8	89119 85.292	94638 73.335	17032 056.03
−9	16787 571.28	17350 592.23	37807 393.57
−10	32285 593.57	31970 742.95	84146 454.83

From Table X-A,

$$X/K_0 = 97{,}871.158,$$

$$X/K_{-10} = 32{,}285{,}594.$$

Thus

$$(X/K_0)\ 0.9116 = 89{,}221.305$$

$$(X/K_{-10})\ 8.3488 \times 10^{-6} = \underline{269.546}$$

$$X = 88{,}951.759$$

Similarly, $Y = 80{,}895$, $Z = 28{,}294.432$.

Also, $X + Y + Z = 198{,}141.196$ and

$$x = 0.44893,\ y = 0.40827.$$

The total reflectance of the sample for Planckian radiation at 2842°K is

$$\rho/\rho_0 = \frac{Y}{89030.4} = 0.9086$$

where the denominator is the Y-value for the incident radiation (Table V). The values of X, Y, Z would be in absolute units (königs/m^2, youngs/m^2, and priests/m^2) if the reflectance curve gave true reflectances. Unfortunately, the curves obtained with recording

TABLE X-B

Coefficients for analytic representation by power functions, illuminant B′ (T = 7000°K)

m	$X/K_m = (G^m)_x$	$Y/K_m = (G^m)_y$	$Z/K_m = (G^m)_z$
0	18412 947.38	19073 137.52	22737 457.35
1	10366 452.66	10600 594.17	10310 063.44
2	59183 99.463	59226 11.023	46877 25.665
3	34198 54.940	33263 89.454	21372 12.357
4	19965 89.838	18780 67.280	97705 9.5833
5	11759 82.397	10659 28.458	44790 4.6469
6	69791 7.3894	60816 8.6704	20589 4.3912
7	41692 6.4500	34881 8.0869	94907. 64391
8	25050 5.2082	20111 8.7997	43869. 11915
9	15128 6.6549	11656 9.4955	20333. 91569
10	91790. 40904	67919. 73482	9451.2 85022
−1	33232 976.53	34497 397.90	50280 623.83
−2	61071 574.79	62721 927.71	11148 9617.3
−3	11447 4305.3	11463 4984.9	24787 9164.2
−4	21916 1863.9	21060 7976.6	55260 4344.9
−5	42887 7903.1	38894 3716.8	12352 47828.
−6	85787 5031.2	72201 5985.7	27685 75112.
−7	17527 87105.	13472 49964.	62218 25423.
−8	36531 57127.	25268 73644.	14019 59734^1
−9	77528 87727.	47637 06843.	31674 25385^1
−10	16719 72348^1	90266 04971.	71750 89482^1

Note: The superscripts refer to the number of zeros before the decimal point. For example, 14019 59734^1—14,019,597,340.

spectrophotometers give relative reflectance $\rho(\lambda)/\rho_0$, where ρ_0 is the true reflectance of the magnesium-oxide comparison surface used in the spectrophotometer. The distinction is a minor one, however, and is usually neglected.

7. Polynomial Representation

When a reflectance or transmittance curve has no obvious similarity to any one of the power functions, there is an advantage in having a general method of

approximating the curve by a polynomial. Such a method was developed by Rutledge.[11] The reflectance curve is represented by

$$\rho(\lambda)/\rho_0 = K_0 + K_1\lambda + K_2\lambda^2 + \cdots + K_6\lambda^6 \tag{28}$$

TABLE XI

Factors for 7-term polynomial representation

k	λ_k (micron)	C_0^k (Factor for K_0)	C_1^k (for K_1)	C_2^k (for K_2)	C_3^k (for K_3)	C_4^k (for K_4)	C_5^k (for K_5)	C_6^k (for K_6)
0	0.40	+3003. 0000	−3205 4.333	+1419 02.22	−3336 50.00	+4388 88.88	−3066 66.66	+8888 8.888
1	0.45	−1601 6.000	+1754 05.33	−7944 13.33	+1906 533.3	−2553 333.3	+1813 333.3	−5333 33.33
2	0.50	+3603 6.000	−4026 70.00	+1859 116.6	−4542 783.3	+6190 000.0	−4466 666.6	+1333 333.3
3	0.55	−4368 0.000	+4960 26.66	−2327 777.7	+5778 666.6	−8004 444.4	+5866 666.6	−1777 777.7
4	0.60	+3003 0.000	−3455 68.33	+1644 433.3	−4139 883.3	+5823 333.3	−4333 333.3	+1333 333.3
5	0.65	−1108 8.000	+1290 16.00	−6213 46.66	+1584 133.3	−2260 000.0	+1706 666.6	−5333 33.33
6	0.70	+1716. 0000	−2015 5.333	+9808 5.555	−2530 16.66	+3655 55.55	−2800 00.00	+8888 8.888

Note: When the same integer appears two or more times at the end of a number, a repeating decimal is indicated.

TABLE XII

Polynomial approximation for wall board no. 221

(See reference 10)

λ	$\rho(\lambda)/\rho_0$	C_0^k Table (XI)	Product
0.40μ	0.225	+3003	+675.675
0.45	0.260	−1601 6	−4164.160
0.50	0.313	+3603 6	+11279.268
0.55	0.405	−4368 0	−17690.400
0.60	0.555	+3003 0	+16666.650
0.65	0.752	−1108 8	−8338.176
0.70	0.970	+1716	+1664.520
			K_0 = +93.3770

and the coefficients K_m are evaluated by taking 7 equally spaced ordinates of the curve. If $\lambda_0 = 0.40\mu$, $\lambda_1 = 0.45$, $\cdots$ $\lambda_6 = 0.70$, then

$$K_m = C_m^k\,\rho(\lambda_k), \qquad (m, k = 0, 1, 2, \cdots 6) \tag{29}$$

where the Einstein summation convention is employed. The constants C_m^k are obtained [11] from Table XI. An example of calculation of K_0 is given in Table XII, the other coefficients being obtained in a similar manner.

The next step is to obtain the trichromatic specification (X, Y, Z) for the reflected radiation by means of Eq. (17). The values of Table X are used and summations of seven terms are made. Let

$$X/K_m = (G^m)_x\,, \qquad Y/K_m = (G^m)_y\,, \qquad Z/K_m = (G^m)_z\,.$$

TABLE XIII

Example of integration, no. 151 green tile, Planckian radiation, T = 2848 K

m	K_m	X	Y	Z
0	−468. 71400	−45873 581.75	−41729 809.62	−1498 929.50
1	+5547. 26033	32292 6782.0	28200 8634.4	8131 320.70
2	−27077. 24666	−94308 5175.4	−79004 0285.9	−18393 8440.9
3	+69384. 50000	14597 11306.	11728 88686.	2965 6040.0
4	−99603. 33333	−12607 47844.	−97204 1612.3	−14608 5437.3
5	+74946. 66666	57569 9540.0	42625 3494.3	51286 356.57
6	−23200. 00000	−10858 0843.2	−77289 919.03	−74275 85.140
		X = 50184 königs/m^2	Y = 49187 youngs/m^2	Z = 13324.5 priests/m^2

$$X + Y + Z = 112695.5$$

$$x = 0.44531, \quad y = 0.43646, \quad z = 0.11823$$

$$\rho/\rho_0 = \frac{49187}{89030.4} = 0.55247$$

By selected ordinates,

$$x = 0.4455, \quad y = 0.4364, \quad \rho/\rho_0 = 0.552$$

TABLE XIV

Direct integration method, values of F^k, to be used in Eq. (31)

k	λ	For X	For Y	For Z
0	0.40μ	273.95	75.019	1102.9
1	0.45	3994.1	−334.56	22,033.7
2	0.50	537.14	11,173.9	8887.2
3	0.55	16853.	37,303.5	−123.04
4	0.60	56,501.	34,274.6	−181.43
5	0.65	18,530.	5685.9	+96.956
6	0.70	1161.5	834.58	−16.608

TABLE XV

Example of direct computation of X, Y, Z, no. 151 green ceramic tile, Planckian radiation (T = 2848°K)

k	λ_k	$\rho(\lambda_k)/\rho_0$	$\rho(\lambda_k)F^k/\rho_0$ For X	$\rho(\lambda_k)F^k/\rho_0$ For Y	$\rho(\lambda_k)F^k/\rho_0$ For Z
0	0.40μ	0.220	60.	16.	242.
1	0.45	0.381	1521.	−127.	8394.
2	0.50	0.542	291.	6056.	4816.
3	0.55	0.601	10128.	22419.	−73.
4	0.60	0.527	29776.	18062.	−95.
5	0.65	0.430	7967.	2444.	+41.
6	0.70	0.370	429.	308.	−6.
			X = 50175. königs/m^2	Y = 49181. youngs/m^2	Z = 13320. priests/m^2

$$X + Y + Z = 112676$$

$$x = 0.44530, \quad y = 0.43648, \quad z = 0.11822$$

$$\rho/\rho_0 = \frac{49181}{89030.4} = 0.5524$$

By selected ordinates,

$$x = 0.4455, \quad y = 0.4364, \quad \rho/\rho_0 = 0.552$$

Then each coordinate of Eq. (17) is of the form:

$$L = G^m \left(C^k_m \rho(\lambda_k)\right) = G^m K_m . \tag{30}$$

Thus the integrals are easily evaluated after the coefficients K_m have been found. An example is given in Table XIII.

An alternative method of evaluating the integrals is by performing the two summations of Eq. (30) in the reverse order:

$$L = \rho(\lambda_k) \left(C^k_m G^m\right) = \rho(\lambda_k) F^k. \tag{31}$$

The new constants,

$$F^k = C^k_m G^m \tag{32}$$

are listed in Table XIV for incident Planckian radiation at 2842°K. By this *direct method of integration*, seven values of reflectance are read from the experimental curve. Each of these values is multiplied by the corresponding values of F^k and the products are added, Eq. (31). A sample computation is given in Table XV.

8. Summary

The purpose of the paper has been to present, in as convenient a form as possible, the equations and tables needed in the practical application of analytic methods to photometry and colorimetry. The new method replaces the customary numerical *tables* of trichromatic weighting factors by weighting *functions* defined by mathematical expressions. In this way, ambiguities in interpolation and extrapolation are eliminated, integrations can be performed analytically, and advantage can be taken of the compactness and simplicity of analytic methods in comparison with the purely numerical procedures that have been required in the past.

The use of gamma functions allows all necessary integrations to be performed very simply. Though the spectral locus specfied by the analytic method differs from that specified by the standard C.I.E. data, the *integrations* obtained by the new method differ generally by less than a minimum-perceptible step from those obtained in the conventional manner. This conclusion applies also to colorimetric integrations involving reflectance and transmittance functions, which we also represent by mathematical expressions. Thus in the great majority of practical cases, the new method and the standard C.I.E. method can be used interchangeably. The two are supplementary and the one to be employed in a specific case depends on the relative advantages of the two for that particular case.

In conclusion, we are happy to express our thanks to Messrs. J. M. Margolskee, A. L. Keller, and R. Kramer for their painstaking care in computing the tables.

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ON THE TRANSFORMATION OF THE EQUATIONS OF DYNAMICS

By T. Y. Thomas

1. Introduction

Due to the present interest in non-linear mechanics we have been led to treat again the problem of the transformation of dynamical systems which was studied, without complete success, by Levi-Civita, Painlevé, Staeckel and others, around the beginning of the present century.[1] The problem can be formulated briefly as follows:

Consider two Lagrangian systems D and E referred to generalized coordinates $x^1, \cdots, x^n$ and $y^1, \cdots, y^n$ respectively. Under what conditions will there exist a (1, 1) correspondence $x \leftrightarrow y$ between the coordinates of these systems by means of which all trajectories of D will be mapped into trajectories of E and conversely? Assuming the existence of such a correspondence we can refer the system E to the coordinates x. When this is done, D and E will admit identical trajectories (considered without regard to their time parameterizations). We are thus led to consider the conditions under which two dynamical systems will have identical trajectories when these systems are referred to the same set of generalized coordinates.

To state the problem more explicitly let the Lagrangian systems D and E be represented by

$$D: \frac{d^2x^\alpha}{dt^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} = Q^\alpha, \qquad (Q \neq 0), \tag{1.1}$$

$$E: \frac{d^2x^\alpha}{d\tau^2} + \Lambda^\alpha_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = R^\alpha, \qquad (R \neq 0), \tag{1.2}$$

where Q and R are the generalized force vectors which are assumed to depend on the coordinates x alone, and the Γ's and Λ's are Christoffel symbols derived respectively from the coefficients $g_{\alpha\beta}(x)$ and $h_{\alpha\beta}(x)$ in the expressions for the kinetic energies of the two systems. The kinetic energies are assumed to be of the form

$$\tfrac{1}{2}g_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} \quad \text{and} \quad \tfrac{1}{2}h_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}.$$

It is assumed implicitly, when not provided for by expressed conditions in the following discussion, that these expressions for the kinetic energies are positive definite forms as required in the dynamical problem.

[1] T. Levi-Civita, *Sulle trasformazioni delle equazioni dinamiche*, Annali di Matematica, Ser. 2, 24, 1896. P. Painlevé, *Sur la transformation des équations de la dynamique*, Journal de Mathématiques, Ser. 4, 10, 1894. See G. Ricci and T. Levi-Civita, *Méthodes de calcul différentiel absolu*, Mathematischen Annalen, 54, 1901, p. 186 for a discussion of investigations of this problem and additional references.

It is easy to show that if one of the force vectors vanishes at a point, the other must also vanish at this point, if the trajectories of the two systems are the same. In the following it will be assumed that neither of the vectors Q and R vanishes at any point of the region under consideration as indicated in the above equations.[2]

In our approach to the problem we have thought of the system D as given and the system E as any other dynamical system (with non-vanishing force vector) all of whose trajectories are trajectories of D; the condition that all trajectories of D are also trajectories of E is later imposed as a separate requirement. On account of the similarity between this problem and the projective theory of the affinely connected space, it is natural that we should attempt to find the relation between the above quantities Γ and Λ, analogous to the relation between the connections of two affinely connected spaces whose paths are in projective correspondence. This has been done (§3) and the relation so determined forms the basis of our development.[3] Differential conditions which are both necessary and sufficient for all trajectories of the system E to be trajectories of the system D are determined. These conditions, together with similar conditions for all trajectories of D to be trajectories of E, give the requirements for the trajectories of D and E to be identical; it turns out that this will be the case only if the geodesics of the two systems correspond and the force vectors are proportional, the factor of proportionality being a simple function of the ratio of the determinants of the coefficients in the expressions for the kinetic energies of the two systems (§7). This dynamical problem is thus reduced to the geometrical problem of the correspondence of geodesics of Riemann spaces, for which specific results have been obtained by Levi-Civita (*loc. cit.* 1).

When the systems D and E are conservative, so that the force vectors Q and R are derivable from potential functions $V(x)$ and $W(x)$, we have deduced the general form of the functions $h_{\alpha\beta}$ and W in terms of the functions $g_{\alpha\beta}$ and V under the assumption that all trajectories of E are trajectories of D; these relations involve the coefficients in a basis of quadratic integrals of energy type[4] for the system D. In case the basis contains only the ordinary energy integral (general case) the relations reduce to those of the type given by Painlevé[5] (*loc.*

[2] If the vectors Q and R vanish identically the problem would revert to the purely geometrical problem of the determination of Riemann spaces with corresponding geodesics.

[3] The lack of this fundamental relation would seem to account for the widely divergent and rather unsystematic procedures employed by the earlier writers in treating this problem.

[4] T. Y. Thomas, *The fundamental theorem on quadratic integrals*, Proc. N. A. S., vol. 32, 1946, pp. 10–15.

[5] No direct derivation of the relationship between the quantities $h_{\alpha\beta}$, W and $g_{\alpha\beta}$, V has been given by Painlevé under the assumption that the system D admits only a single quadratic first integral (energy integral). In this connection we note the following result, attributed to Painlevé by E. T. Whittaker, *Analytical Dynamics*, 4th ed., 1937, p. 261: *If in two dynamical systems in which the kinetic energies are respectively* $\sum a_{ik}\dot{q}_i\dot{q}_k$ *and* $\sum b_{ik}\dot{q}_i\dot{q}_k$ *and the potential energies are respectively* U *and* V, *the trajectories are the same*

cit. 1) for the correspondence of *all* trajectories of the systems D and E (§9); but under this more stringent condition our relations are further specialized (§11). The reason for this discrepancy appears to lie in the fact that Painlevé has allowed imaginary values of the parameters t and τ while we have restricted these parameters to real values in conformity with the proper interpretation of the dynamical equations.

Dynamical systems of two degrees of freedom are excluded since such systems require a special treatment in certain stages of the theory. However our investigation indicates that even stronger results can be obtained for systems of two degrees of freedom than are given in the present paper for systems of more than two degrees of freedom. It is our intention to publish these results separately.

2. Parameter transformations

Assume all trajectories of E to be trajectories of D. Let C be any trajectory of E so that (1.2) is satisfied when C is referred to the parameter τ and (1.1) is satisfied when C is referred to the parameter t. We assume that the parameters t and τ are real and that the transformations $t \rightarrow \tau$ and $\tau \rightarrow t$ are continuous and have continuous first and second derivatives. Making the transformation $\tau \rightarrow t$ the equations (1.2) become

$$\frac{d^2x^\alpha}{dt^2} + \Lambda^\alpha_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} = R^\alpha\left(\frac{d\tau}{dt}\right)^2 - \frac{d^2t}{d\tau^2}\left(\frac{d\tau}{dt}\right)^2\frac{dx^\alpha}{dt}. \tag{2.1}$$

Subtracting corresponding members of (1.1) and (2.1) we have

$$\Phi^\alpha_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} = R^\alpha\left(\frac{d\tau}{dt}\right)^2 - Q^\alpha - \frac{d^2t}{d\tau^2}\left(\frac{d\tau}{dt}\right)^2\frac{dx^\alpha}{dt}, \tag{2.2}$$

where,

$$\Phi^\alpha_{\mu\nu} = \Lambda^\alpha_{\mu\nu} - \Gamma^\alpha_{\mu\nu}.$$

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curves though described with different velocities, so that the relations between the coordinates $(q_1, q_2, \cdots, q_n)$ *are the same in the two problems, show that*

$$V = \frac{\alpha U + \beta}{\gamma U + \delta},$$

where $\alpha, \beta, \gamma, \delta$ *are constants, and that*

$$\sum b_{ik}\,dq_i\,dq_k = (\gamma U + \delta)\sum a_{ik}\,dq_i\,dq_k.$$

This is not correct. For example, let $g_{\alpha\beta} = \delta_{\alpha\beta}$, $V = -\delta_{\alpha\beta}x^\alpha x^\beta$ determine the system D and $h_{\alpha\beta} = c_{\alpha\beta}$, $W = -c_{\alpha\beta}x^\alpha x^\beta$ the system E, where the c's are any constants subject to the condition that $\det.\,|c_{\alpha\beta}| \neq 0$. Then

$$D: \frac{d^2x^\alpha}{dt^2} = -\delta^{\alpha\mu}V_{,\mu} = -V_{,\alpha} = 2x^\alpha,$$

$$E: \frac{d^2x^\alpha}{d\tau^2} = -c^{\alpha\mu}W_{,\mu} = 2c^{\alpha\mu}c_{\mu\beta}x^\beta = 2x^\alpha.$$

Hence systems D and E have identical trajectories but the above relations do not necessarily hold.

Now multiply (2.2) by dx^β/dt and then interchange α and β and subtract to obtain

$$\left(\Phi^\alpha_{\mu\nu}\frac{dx^\beta}{dt} - \Phi^\beta_{\mu\nu}\frac{dx^\alpha}{dt}\right)\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} = \left(R^\alpha\frac{dx^\beta}{dt} - R^\beta\frac{dx^\alpha}{dt}\right)\left(\frac{d\tau}{dt}\right)^2 - \left(Q^\alpha\frac{dx^\beta}{dt} - Q^\beta\frac{dx^\alpha}{dt}\right). \tag{2.3}$$

Next multiply (2.3) by the quantity $Q^\gamma dx^\delta/dt - Q^\delta dx^\gamma/dt$ and from the resulting equations subtract those obtained by simultaneous interchange of the indices α, γ and β, δ. This gives a system of equations which can be written

$$\begin{aligned}\Big[\Big(Q^\gamma\frac{dx^\delta}{d\tau} - Q^\delta\frac{dx^\gamma}{d\tau}\Big)\Big(\Phi^\alpha_{\mu\nu}\frac{dx^\beta}{d\tau} - \Phi^\beta_{\mu\nu}\frac{dx^\alpha}{d\tau}\Big) \\ - \Big(Q^\alpha\frac{dx^\beta}{d\tau} - Q^\beta\frac{dx^\alpha}{d\tau}\Big)\Big(\Phi^\gamma_{\mu\nu}\frac{dx^\delta}{d\tau} - \Phi^\delta_{\mu\nu}\frac{dx^\gamma}{d\tau}\Big)\Big]\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \\ = \Big(Q^\gamma\frac{dx^\delta}{d\tau} - Q^\delta\frac{dx^\gamma}{d\tau}\Big)\Big(R^\alpha\frac{dx^\beta}{d\tau} - R^\beta\frac{dx^\alpha}{d\tau}\Big) \\ - \Big(Q^\alpha\frac{dx^\beta}{d\tau} - Q^\beta\frac{dx^\alpha}{d\tau}\Big)\Big(R^\gamma\frac{dx^\delta}{d\tau} - R^\delta\frac{dx^\gamma}{d\tau}\Big).\end{aligned} \tag{2.4}$$

Now the left member of this equation is a form of the fourth degree in the $dx^\alpha/d\tau$ while the right member is a form of the second degree in these quantities; also the $dx^\alpha/d\tau$ can be assigned arbitrary initial values. It follows that the two members of this equation must vanish separately. We thus have the following two systems of equations

$$\begin{aligned}\Big[R^\gamma Q^\alpha\delta^\delta_\mu\delta^\beta_\nu - R^\gamma Q^\beta\delta^\delta_\mu\delta^\alpha_\nu - R^\delta Q^\alpha\delta^\gamma_\mu\delta^\beta_\nu + R^\delta Q^\beta\delta^\gamma_\mu\delta^\alpha_\nu \\ - R^\alpha Q^\gamma\delta^\beta_\mu\delta^\delta_\nu + R^\alpha Q^\delta\delta^\beta_\mu\delta^\gamma_\nu + R^\beta Q^\gamma\delta^\alpha_\mu\delta^\delta_\nu - R^\beta Q^\delta\delta^\alpha_\mu\delta^\gamma_\nu\Big]\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0,\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}\Big[Q^\gamma\Phi^\alpha_{\mu\nu}\delta^\delta_\sigma\delta^\beta_\eta - Q^\gamma\Phi^\beta_{\mu\nu}\delta^\delta_\sigma\delta^\alpha_\eta - Q^\delta\Phi^\alpha_{\mu\nu}\delta^\gamma_\sigma\delta^\beta_\eta + Q^\delta\Phi^\beta_{\mu\nu}\delta^\gamma_\sigma\delta^\alpha_\eta - Q^\alpha\Phi^\gamma_{\mu\nu}\delta^\beta_\sigma\delta^\delta_\eta \\ + Q^\alpha\Phi^\delta_{\mu\nu}\delta^\beta_\sigma\delta^\gamma_\eta + Q^\beta\Phi^\gamma_{\mu\nu}\delta^\alpha_\sigma\delta^\delta_\eta - Q^\beta\Phi^\delta_{\mu\nu}\delta^\alpha_\sigma\delta^\gamma_\eta\Big]\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau}\frac{dx^\eta}{d\tau} = 0.\end{aligned} \tag{2.6}$$

3. Algebraic relations

When the bracket expression in (2.5) is made symmetric in the indices μ and ν it must vanish. This gives the following set of algebraic relations between the quantities Q^α and R^α, namely

$$\begin{aligned}&R^\gamma Q^\alpha\delta^\delta_\mu\delta^\beta_\nu + R^\gamma Q^\alpha\delta^\delta_\nu\delta^\beta_\mu - R^\gamma Q^\beta\delta^\delta_\mu\delta^\alpha_\nu - R^\gamma Q^\beta\delta^\delta_\nu\delta^\alpha_\mu \\ &-R^\delta Q^\alpha\delta^\gamma_\mu\delta^\beta_\nu - R^\delta Q^\alpha\delta^\gamma_\nu\delta^\beta_\mu + R^\delta Q^\beta\delta^\gamma_\mu\delta^\alpha_\nu + R^\delta Q^\beta\delta^\gamma_\nu\delta^\alpha_\mu \\ &-R^\alpha Q^\gamma\delta^\beta_\mu\delta^\delta_\nu - R^\alpha Q^\gamma\delta^\beta_\nu\delta^\delta_\mu + R^\alpha Q^\delta\delta^\beta_\mu\delta^\gamma_\nu + R^\alpha Q^\delta\delta^\beta_\nu\delta^\gamma_\mu \\ &+ R^\beta Q^\gamma\delta^\alpha_\mu\delta^\delta_\nu + R^\beta Q^\gamma\delta^\alpha_\nu\delta^\delta_\mu - R^\beta Q^\delta\delta^\alpha_\mu\delta^\gamma_\nu - R^\beta Q^\delta\delta^\alpha_\nu\delta^\gamma_\mu = 0.\end{aligned} \tag{3.1}$$

Now put $\beta = \mu$ and $\delta = \nu$ in these equations and sum are the repeated indices. Under the assumption that the dynamical systems D and E have $n > 2$ degrees of freedom we thus obtain the following relations: $R^\gamma Q^\alpha = R^\alpha Q^\gamma$. In other words the quantities R^α and Q^α are proportional[6] so that we can write

$$R^\alpha = \psi Q^\alpha. \tag{3.2}$$

Since neither of the vectors Q and R vanishes by assumption it follows that the function ψ does not vanish. The right member of (2.4) is immediately seen to vanish as a result of the substitution $R^\alpha = \psi Q^\alpha$; hence (3.2) gives all conditions obtainable from (2.5).

We now treat (2.6) in a similar manner. This leads to a set of algebraic relations, corresponding to (3.1), which we shall merely represent by writing

$$Q^\gamma \Phi^\alpha_{\mu\nu\delta\epsilon\eta} + \cdots = 0. \tag{3.3}$$

Putting $\beta = \epsilon$ and $\delta = \eta$ in (3.3), and summing on the repeated indices, we are led to the following conditions

$$\begin{aligned} n(Q^\alpha \Phi^\gamma_{\mu\nu} - Q^\gamma \Phi^\alpha_{\mu\nu}) + \Phi^\sigma_{\mu\sigma}(Q^\gamma \delta^\alpha_\nu - Q^\alpha \delta^\gamma_\nu) + \Phi^\sigma_{\nu\sigma}(Q^\gamma \delta^\alpha_\mu - Q^\alpha \delta^\gamma_\mu) \\ + Q^\sigma(\Phi^\alpha_{\mu\sigma}\delta^\gamma_\nu - \Phi^\gamma_{\mu\sigma}\delta^\alpha_\nu) + Q^\sigma(\Phi^\alpha_{\nu\sigma}\delta^\gamma_\mu - \Phi^\gamma_{\nu\sigma}\delta^\alpha_\mu) = 0. \end{aligned} \tag{3.4}$$

The expression in the left member of (3.4) is clearly symmetric in μ, ν and skew-symmetric in α, γ. Now multiply these equations by $g_{\alpha\beta}Q^\beta$; then dividing by the quantity $ng_{\alpha\beta}Q^\alpha Q^\beta$ we have a system of the form

$$\Phi^\gamma_{\mu\nu} = \delta^\gamma_\mu p_\nu + \delta^\gamma_\nu p_\mu + r_{\mu\nu}Q^\gamma + s^\gamma_\mu Q_\nu + s^\gamma_\nu Q_\mu, \tag{3.5}$$

where we have put $Q_\mu = g_{\mu\beta}Q^\beta$. We do not make use of the explicit expressions for p_μ and $r_{\mu\nu}$. The expression for s^γ_μ is

$$s^\gamma_\mu = \frac{\Phi^\gamma_{\mu\sigma}Q^\sigma}{ng_{\alpha\beta}Q^\alpha Q^\beta}. \tag{3.6}$$

Now multiply (3.5) by Q^ν and use (3.6) to obtain

$$(n-1)Q_\nu Q^\nu s^\gamma_\mu = \delta^\gamma_\mu p_\nu Q^\nu + (p_\mu + r_{\mu\nu}Q^\nu)Q^\gamma + s^\gamma_\nu Q^\nu Q_\mu. \tag{3.7}$$

Multiplying this by Q^μ we are led to an equation of the form $s^\gamma_\mu Q^\mu = wQ^\gamma$; making this substitution in the right member of (3.7) the resulting equation has the form

$$s^\gamma_\mu = e\delta^\gamma_\mu + f_\mu Q^\gamma.$$

[6] To deduce $R^\alpha = \psi Q^\alpha$ formally from $R^\gamma Q^\alpha = R^\alpha Q^\gamma$ multiply this equation through by Q^γ and sum on the repeated index γ. This gives $\left(\sum Q^\gamma Q^\gamma\right)R^\alpha = \left(\sum R^\gamma Q^\gamma\right)Q^\alpha$. Hence $R^\alpha = \psi Q^\alpha$, where

$$\psi = \frac{\sum R^\gamma Q^\gamma}{\sum Q^\gamma Q^\gamma},$$

since $\sum Q^\gamma Q^\gamma \neq 0$ by hypothesis.

When these expressions for the s_μ^γ are substituted into the right member of (3.5) we find that the Φ's are given by a set of equations of the form

$$\Phi_{\mu\nu}^\gamma = \delta_\mu^\gamma \phi_\nu + \delta_\nu^\gamma \phi_\mu + A_{\mu\nu} Q^\gamma, \tag{3.8}$$

in which the functions $A_{\mu\nu}$ are symmetric in the indices μ and ν. Substituting these values of the Φ's into the left member of (2.4) the resulting expression vanishes identically. Hence (2.6) cannot lead to conditions on the Φ's beyond those given by (3.8) in which the ϕ_μ and the $A_{\mu\nu}(= A_{\nu\mu})$ are arbitrary functions.

4. Parameter transformations (continued)

The above algebraic relations (3.2) and (3.8) lead to certain explicit conditions on the parameter transformation $t \leftrightarrow \tau$ which we shall now determine. Substituting from (3.2) and (3.8) into (2.2) the resulting equation can be written

$$\left[\psi \left(\frac{d\tau}{dt}\right)^2 - 1 - A_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}\right] Q^\alpha = \left[\frac{d^2 t}{d\tau^2} \left(\frac{d\tau}{dt}\right)^2 + 2\phi_\nu \frac{dx^\nu}{dt}\right] \frac{dx^\alpha}{dt}. \tag{4.1}$$

If the initial values of the $dx^\alpha/d\tau$, and hence of dx^α/dt, do not all vanish and if these initial values are not proportional to the Q^α, then (4.1) breaks up into the following two equations

$$\psi \left(\frac{d\tau}{dt}\right)^2 = 1 + A_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}, \tag{4.2}$$

$$\frac{d^2 t}{d\tau^2} \left(\frac{d\tau}{dt}\right)^2 + 2\phi_\nu \frac{dx^\nu}{dt} = 0. \tag{4.3}$$

For, if neither (4.2) nor (4.3) is satisfied Q^α proportional to dx^α/dt; if (4.2) is satisfied but (4.3) is not, all dx^α/dt vanish; if (4.2) is not satisfied but (4.3) is satisfied, then all Q^α vanish contrary to our assumption. Equations equivalent to (4.2) and (4.3) are

$$\left(\frac{dt}{d\tau}\right)^2 = \psi - A_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}, \tag{4.4}$$

$$\frac{\dfrac{d^2\tau}{dt^2}}{\dfrac{d\tau}{dt}} = 2\phi_\nu \frac{dx^\nu}{dt}. \tag{4.5}$$

It will be shown in the following section that ϕ_ν is the gradient of a scalar function ϕ. We can put $4\phi = \log \omega$. Anticipating this result, equation (4.5) can be integrated, giving

$$\frac{d\tau}{dt} = Be^{2\phi} = B\omega^{\frac{1}{2}}, \tag{4.6}$$

where the constant of integration B may depend on the trajectory.

5. Differential relations

Let us now differentiate (4.4) with respect to τ and then eliminate the second derivatives $d^2x^\alpha/d\tau^2$ from the resulting equations by means of (1.2). When we furthermore eliminate the derivative $d^2t/d\tau^2$ by (4.3) and the quantity $(dt/d\tau)^2$ by (4.4) the following equation is obtained

$$(A_{\alpha\beta;\gamma} + 4A_{\alpha\beta}\phi_\gamma)\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} + (2A_{\alpha\beta}R^\beta - \psi_{,\alpha} - 4\psi\phi_\alpha)\frac{dx^\alpha}{d\tau} = 0,$$

where the "semi-colon" denotes covariant differentiation based on the Λ. Hence making the substitution $R^\beta = \psi Q^\beta$ we have[7]

$$2A_{\alpha\beta}Q^\beta = (\log\psi)_{,\alpha} + 4\phi_\alpha, \tag{5.1}$$

$$A_{\alpha\beta;\gamma} + A_{\beta\gamma;\alpha} + A_{\gamma\alpha;\beta} + 4(A_{\alpha\beta}\phi_\gamma + A_{\beta\gamma}\phi_\alpha + A_{\gamma\alpha}\phi_\beta) = 0. \tag{5.2}$$

Now put $\gamma = \nu$ in (3.8) and sum on the repeated indices. This gives

$$\Phi^\nu_{\mu\nu} = (n+1)\phi_\mu + A_{\mu\nu}Q^\nu,$$

or,

$$4\phi_\mu = \frac{4\Phi^\alpha_{\mu\alpha}}{n+1} - \frac{4A_{\mu\alpha}Q^\alpha}{n+1} = 2A_{\mu\alpha}Q^\alpha - (\log\psi)_{,\mu},$$

when use is made of (5.1). Hence

$$\left(\frac{2n+6}{n+1}\right)A_{\mu\alpha}Q^\alpha = \frac{4\Phi^\alpha_{\mu\alpha}}{n+1} + (\log\psi)_{,\mu}. \tag{5.3}$$

But,

$$\Phi^\alpha_{\mu\alpha} = \frac{1}{2}\left(\log\frac{h}{g}\right)_{,\mu},$$

where $h = \det.\,|\,h_{\alpha\beta}\,|$ and $g = \det.\,|\,g_{\alpha\beta}\,|$. Hence (5.3) becomes

$$A_{\mu\alpha}Q^\alpha = \left[\frac{1}{n+3}\log\left(\frac{h}{g}\right) + \frac{n+1}{2n+6}\log\psi\right]_{,\mu}. \tag{5.4}$$

In other words the quantities $A_{\mu\alpha}Q^\alpha$ are the partial derivatives of the above expression in brackets. It now follows from (5.1) and (5.4) that ϕ_μ *is the gradient of a function* ϕ. This fact was used in §4 in the derivation of equation (4.6). We find

$$\phi = \frac{1}{2n+6}\log\left(\frac{h}{g\psi}\right) + \text{const.} \tag{5.5}$$

It is clearly permissible to take the additive constant to be zero in this equation. Doing this, and making the substitution $4\phi = \log\omega$, as in §4, we have

$$\omega = \left(\frac{h}{g\psi}\right)^{2/(n+3)}. \tag{5.6}$$

[7] In place of $\log\psi$ in (5.1) we could equally well have written $\log(-\psi)$. Hence no assumption is here implied concerning the algebraic sign of the function ψ.

When the quantity ω is introduced into (5.1) and (5.2) these equations become

$$2A_{\alpha\beta}Q^{\beta} = (\log \omega\psi)_{,\alpha}, \text{ and} \tag{5.7}$$

$$(\omega A_{\alpha\beta})_{;\gamma} + (\omega A_{\beta\gamma})_{;\alpha} + (\omega A_{\gamma\alpha})_{;\beta} = 0. \tag{5.8}$$

6. The quadratic first integral

On account of (4.6) the equation (4.4) becomes

$$\omega A_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau} - \omega\psi = -\frac{1}{B^2}. \tag{6.1}$$

Hence (6.1) is satisfied along any trajectory of E which is subject to the condition under which (4.2) and (4.3) were derived, i.e. that not all $dx^{\alpha}/d\tau$ vanish and the values of $dx^{\alpha}/d\tau$ are not proportional to Q^{α}. Subject to this restriction we can select the values of the $dx^{\alpha}/d\tau$ arbitrarily small; then, since $\omega > 0$, and the right member of (6.1) is negative, it *follows that* $\psi > 0$. *Also we must have* $A_{\mu\nu}\xi^{\mu}\xi^{\nu} \leqq 0$ *for arbitrary (real) values of the* ξ's. For, suppose $A_{\mu\nu}\eta^{\mu}\eta^{\nu} > 0$. We can then find a set of values of $\xi^{\mu} = \xi_0^{\mu}$ such that (a) the ξ_0^{μ} will not all vanish, (b) the ξ_0^{μ} will not be proportional to the Q^{μ}, and (c) the condition $A_{\mu\nu}\xi_0^{\mu}\xi_0^{\nu} > 0$ will be satisfied. Hence if we take $dx^{\alpha}/d\tau = k\xi_0^{\alpha}$ initially the equation (6.1) will hold along the trajectory. But by choosing k sufficiently large the first term of (6.1) will be positive and numerically larger than the second term in the left member of this equation; this will be in contradiction with the fact that the right member of (6.1) is negative.

We now show that (6.1) *is a first integral of the system* E. This implies that for a suitable selection of the (real) constant B, the equation (6.1) holds along any trajectory of E without regard to the above restriction on the derivatives $dx^{\alpha}/d\tau$. In this connection we note the conditions for

$$H_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau} + U = \text{const.} \tag{6.2}$$

to be a first integral of E. To obtain these conditions we differentiate (6.2) with respect to τ and eliminate the second derivatives $d^2x^{\alpha}/d\tau^2$ by the substitution (1.2); when we then impose the condition that the initial values of the $dx^{\alpha}/d\tau$ are arbitrary we obtain the required equations. These are

$$U_{,\mu} + 2H_{\mu\nu}R^{\nu} = 0, \tag{6.3}$$

$$H_{\mu\nu;\eta} + H_{\nu\eta;\mu} + H_{\eta\mu;\nu} = 0. \tag{6.4}$$

The conditions (6.3) and (6.4) are both necessary and sufficient for (6.2) to be a first integral of E. Now taking $H_{\mu\nu} = \omega A_{\mu\nu}$ and $U = -\omega\psi$ it follows immediately from (5.7) and (5.8) that (6.3) and (6.4) are satisfied. Hence

$$\omega A_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau} - \omega\psi = \text{const.} \tag{6.5}$$

is an integral of the system E. Then since $\omega > 0$, $\psi > 0$ and $A_{\mu\nu}\xi^\mu\xi^\nu \leqq 0$ for arbitrary ξ's, it follows that the constant in the right member of (6.5) is necessarily negative. Hence (6.1) is a first integral of E.

Introducing the parameter t in (6.1) by means of the substitution (4.6) the resulting equation can be written.

$$\left(\frac{A_{\mu\nu}}{\omega\psi}\right)\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} + \frac{1}{\omega\psi} = B^2. \tag{6.6}$$

This suggests that

$$\left(\frac{A_{\mu\nu}}{\omega\psi}\right)\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} + \frac{1}{\omega\psi} = \text{const.} \tag{6.7}$$

is an integral of the system D. Corresponding to (6.3) and (6.4) the required conditions for this are

$$\left(\frac{1}{\omega\psi}\right)_{,\mu} + 2\left(\frac{A_{\mu\nu}}{\omega\psi}\right)Q^\nu = 0, \quad \text{and} \tag{6.8}$$

$$\left(\frac{A_{\mu\nu}}{\omega\psi}\right)_{,\eta} + \left(\frac{A_{\nu\eta}}{\omega\psi}\right)_{,\mu} + \left(\frac{A_{\eta\mu}}{\omega\psi}\right)_{,\nu} = 0, \tag{6.9}$$

where the "comma" denotes covariant differentiation based on Γ. But (6.8) is satisfied in consequence of (5.7) and (6.9) is seen to be satisfied on account of (5.8) combined with (3.8) and (5.7). *Hence* (6.7) *is a first integral of the system* D. We observe that the constant in the right member of (6.7) can be negative provided the condition $A_{\mu\nu}\xi^\mu\xi^\nu < 0$ can be realized; in such a case the integral (6.7) is not equivalent to (6.6).

7. General correspondence theorems

We now prove the following result. *All trajectories of the dynamical system E will be trajectories of the dynamical system D if, and only if, the following conditions are satisfied:*

(a) $$\Phi^\alpha_{\mu\nu} = \delta^\alpha_\mu\phi_\nu + \delta^\alpha_\nu\phi_\mu + A_{\mu\nu}Q^\alpha,$$

(b) $$R^\alpha = \psi Q^\alpha, \qquad \psi > 0,$$

(c) $$\frac{A_{\mu\nu}}{\omega\psi}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} + \frac{1}{\omega\psi} = \text{const.}$$

is a first integral of D where $\omega = (h/g\psi)^{2/n+3}$ *and* $A_{\mu\nu}\xi^\mu\xi^\nu \leqq 0$ *for all* ξ.

The necessity of these conditions has been proved. It remains therefore to show their sufficiency. First, since (c) is an integral of D we have (6.8) and (6.9). Hence (5.7) holds since this is equivalent to (6.8). Now contract the indices α and ν in (a) and then eliminate the quantity h/g from the resulting equations by means of the above equation for ω; this leads to the relation $4\phi_\mu = (\log\omega)_{,\mu}$. Expand the terms in (6.9) and then eliminate the components Γ by means of (a); then making use of (5.7) and the equation $4\phi_\mu = (\log\omega)_{,\mu}$ we can deduce (5.8). But from (5.7) and (5.8) it follows that (6.5) is a first

integral of E. From the conditions $\omega > 0$, $\psi > 0$ and $A_{\mu\nu}\xi^\mu\xi^\nu \leqq 0$ we then deduce that (6.1) is a first integral of E as in §6.

Now let C be any trajectory of E, defined as a solution of (1.2). Then (6.1) holds along C for a suitable selection of the constant B. Using this value of B we define the parameter transformation $\tau \leftrightarrow t$ by means of (4.6). From (1.2) we now deduce (2.1) along C and from these equations we eliminate the Λ's by means of (a) and the R^α by (b). By recourse to (4.6), (6.1) and the substitution $4\phi_\mu = (\log \omega)_\mu$ it now readily follows that these equations reduce to (1.1) along C.[8] Hence C is a trajectory of the system D and the proof is complete.

The conditions for all trajectories of D to be trajectories of E can readily be inferred from the above result by making the proper changes in the functions involved. Corresponding to the interchange in the roles of the functions Γ and Λ let us now write (a) in the form

$$\Gamma^\alpha_{\mu\nu} - \Lambda^\alpha_{\mu\nu} = \delta^\alpha_\mu(-\phi_\nu) + \delta^\alpha_\nu(-\phi_\mu) + \left(\frac{-A_{\mu\nu}}{\psi}\right)R^\alpha .$$

Also write $Q^\alpha = (1/\psi)R^\alpha$ in place of (b) and note that $4(-\phi_\mu) = (\log 1/\omega)_\mu$. It is clear, therefore, that if we make the substitutions $\omega \to 1/\omega$, $\psi \to 1/\psi$ and $A_{\mu\nu} \to -A_{\mu\nu}/\psi$ in the first integral (c) and the condition $A_{\mu\nu}\xi^\mu\xi^\nu \leqq 0$, we shall obtain the new form of these requirements corresponding to the problem under consideration. But this gives

$$-\omega A_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + \omega\psi = \text{const.}$$

as a first integral of the system E and $-A_{\mu\nu}\xi^\mu\xi^\nu \leqq 0$ for all ξ.

Now suppose that all trajectories of E are trajectories of D and conversely. Then $A_{\mu\nu}\xi^\mu\xi^\nu \leqq 0$ and $-A_{\mu\nu}\xi^\mu\xi^\nu \leqq 0$ for all ξ which implies $A_{\mu\nu} = 0$. Hence from (c) we have $\psi = c^2/\omega$ where c is an absolute constant. But since $\omega = e^{4\phi}$ we can also write $\psi = c^2e^{-4\phi}$ where now $2(n+1)\phi = \log(h/g)$; this latter relation is obtained by contracting the indices α and ν in (c) with $A_{\mu\nu} = 0$. We thus have the following result.

All trajectories of E will be trajectories of D and conversely if, and only if,

$$(\alpha)\quad \Phi^\alpha_{\mu\nu} = \delta^\alpha_\mu\phi_\nu + \delta^\alpha_\nu\phi_\mu ,$$

$$(\beta)\quad R^\alpha = c^2\left(\frac{g}{h}\right)^{2/(n+1)}Q^\alpha ,$$

where c is a constant, $g = \det.\,|\,g_{\alpha\beta}\,|$ *and* $h = \det.\,|\,h_{\alpha\beta}\,|$. Condition ($\alpha$) is the well known condition for the geodesics of D and E to be the same curves in space. The determination of all dynamical systems E having the same trajectories as a given system D is thus reduced to the solution of the geometrical problem of finding all Riemann spaces whose geodesics are the same as those of a given Rie-

[8] It is thus shown that under the above conditions (a), (b), and (c) we can pass from (1.2) to (1.1) by the transformation (4.6) without the restriction on the curve C which was introduced in §4 in deriving this transformation.

mann space. Corresponding to any solution of this geometrical problem there will be a dynamical system E, with force vector given by (β), such that the systems D and E have identical trajectories.

8. Conservative systems

Suppose that the systems D and E are conservative and denote the potential functions of these systems by $V(x)$ and $W(x)$ respectively. We can then write

$$Q^\alpha = -g^{\alpha\sigma}V_{,\sigma}, \qquad R^\alpha = -h^{\alpha\sigma}W_{,\sigma}.$$

Now D and E admit respectively the following first integrals

$$\tfrac{1}{2}g_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} + V = \text{const.}, \tag{8.1}$$

$$\tfrac{1}{2}h_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} + W = \text{const.}, \tag{8.2}$$

expressing the condition that the total energy along any trajectory is constant. We now treat the problem of finding all quantities $h_{\alpha\beta}$ and W, by which the system E is determined, so that all trajectories of E will be trajectories of the given system D.

In treating this problem all results can be applied which were previously obtained in dealing with the corresponding problem without the conservative condition. First, let us note the following relations, which will be needed in our discussion, namely

$$h_{\alpha\beta}Q^\beta = \frac{h_{\alpha\beta}R^\beta}{\psi} = -\frac{h_{\alpha\beta}h^{\beta\sigma}W_{,\sigma}}{\psi} = -\frac{W_{,\alpha}}{\psi}. \tag{8.3}$$

Now transpose (3.8) to normal coordinates based on the Γ's and evaluate at the origin; the resulting equations can be solved for the quantities $h_{\mu\nu,\alpha}$ to yield

$$h_{\mu\nu,\alpha} = 2h_{\mu\nu}\phi_\alpha + h_{\mu\alpha}\phi_\nu + h_{\nu\alpha}\phi_\mu + (h_{\mu\sigma}Q^\sigma)A_{\nu\alpha} + (h_{\nu\sigma}Q^\sigma)A_{\mu\alpha}. \tag{8.4}$$

Permute the indices μ, ν, α cyclically in (8.4) and add the resulting equations; then making the substitution $4\phi = \log\omega$ and using (8.3) and (6.9) we readily find that

$$\left(\frac{h_{\mu\nu}}{\omega} + \frac{2WA_{\mu\nu}}{\omega\psi}\right)_{,\alpha} + \left(\frac{h_{\nu\alpha}}{\omega} + \frac{2WA_{\nu\alpha}}{\omega\psi}\right)_{,\mu} + \left(\frac{h_{\alpha\mu}}{\omega} + \frac{2WA_{\alpha\mu}}{\omega\psi}\right)_{,\nu} = 0. \tag{8.5}$$

Also

$$\left(\frac{h_{\mu\nu}}{\omega} + \frac{2WA_{\mu\nu}}{\omega\psi}\right)g^{\nu\sigma}V_{,\sigma} = -\frac{h_{\mu\nu}Q^\nu}{\omega} - \frac{2W}{\omega\psi}A_{\mu\nu}Q^\nu \tag{8.6}$$

$$= \frac{W_{,\mu}}{\omega\psi} - \frac{W}{\omega\psi}(\log\omega\psi)_{,\mu} = \left(\frac{W}{\omega\psi}\right)_{,\mu},$$

where use has been made of (5.7) and (8.3). Comparing with (6.3) and (6.4) we see that (8.5) and (8.6) are the conditions for

$$\frac{1}{2}\left(\frac{h_{\mu\nu}}{\omega}+\frac{2WA_{\mu\nu}}{\omega\psi}\right)\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}+\frac{W}{\omega\psi}=\text{const.} \tag{8.7}$$

to be a first integral of the conservative system D.

Now let

$$\tfrac{1}{2}g^{(i)}_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}+V^{(i)}=\text{const.}, \qquad (i=1,\cdots,s), \tag{8.8}$$

be a basis of quadratic integrals of energy type for the system D (*loc. cit.* 4). Then from (6.7) and (8.7) we have

$$\begin{cases} \dfrac{A_{\mu\nu}}{\omega\psi}=\tfrac{1}{2}m_i\,g^{(i)}_{\mu\nu}\,, \\ \dfrac{1}{\omega\psi}=m_i\,V^{(i)}+e\,, \\ \dfrac{h_{\mu\nu}}{\omega}+\dfrac{2W}{\omega\psi}A_{\mu\nu}=c_i\,g^{(i)}_{\mu\nu}\,, \\ \dfrac{W}{\omega\psi}=c_i\,V^{(i)}+d\,, \end{cases} \tag{8.9}$$

where the c_i, m_i, d and e are constants and the summation on the index i is over the range $1,\cdots,s$. The relations (8.9) imply that (6.7) and (8.7) are first integrals of D. From (8.9) we deduce

$$h_{\mu\nu}=\omega(c_i-Wm_i)g^{(i)}_{\mu\nu}\,, \tag{8.10}$$

$$W=\frac{c_i\,V^{(i)}+d}{m_i\,V^{(i)}+e}\,, \tag{8.11}$$

$$A_{\mu\nu}=\frac{1}{2}\,\frac{m_i\,g^{(i)}_{\mu\nu}}{m_k\,V^{(k)}+e}\,. \tag{8.12}$$

From the second equation (8.9) we see that the constants m_i and e must be chosen so that $m_kV^{(k)}+e>0$, since $\omega>0$ and $\psi>0$. The relation (8.11) gives W interms of the quantities $V^{(i)}$ associated with the system D. Eliminating W from the right member of (8.10) by (8.11) we have the $h_{\mu\nu}$ expressed in terms of ω and the invariants $V^{(i)}$ and $g^{(i)}$ of the system D.

We now show that the quantity ω can likewise be expressed in terms of the above invariants. Taking the determinant of both members of (8.10) we have

$$h=\omega^n\,|\,(c_i-Wm_i)g^{(i)}_{\mu\nu}\,|. \tag{8.13}$$

Use (8.13) to eliminate the h in the right member of (5.6) and then eliminate the ψ by means of the second equation (8.9). Solving the resulting equation for ω we have

$$\omega=\left[\frac{g}{(m_k\,V^{(k)}+e)\,|\,(c_i-Wm_i)g^{(i)}_{\mu\nu}\,|}\right]^{2/(n-1)}. \tag{8.14}$$

The condition $R^\alpha = \psi Q^\alpha$ is equivalent to

$$h^{\alpha\sigma}W_{,\sigma} = \psi g^{\alpha\sigma}V_{,\sigma} \quad \text{or} \quad W_{,\mu} = \psi h_{\mu\nu}g^{\nu\sigma}V_{,\sigma}. \tag{8.15}$$

But these conditions are satisfied automatically on account of (8.9). For we have

$$\begin{aligned}\psi h_{\mu\nu}g^{\nu\sigma}V_{,\sigma} &= \psi\omega(c_i - Wm_i)g^{(i)}_{\mu\nu}g^{\nu\sigma}V_{,\sigma}\\ &= \frac{(c_i - Wm_i)g^{(i)}_{\mu\nu}g^{\nu\sigma}V_{,\sigma}}{(m_k V^{(k)} + e)}\\ &= \frac{(c_i - Wm_i)V^{(i)}_{,\mu}}{(m_k V^{(k)} + e)}\end{aligned}$$

when use is made of the conditions expressing the fact that (8.8) are first integrals of D. On the other hand

$$\begin{aligned}W_{,\mu} &= \left(\frac{c_i V^{(i)} + d}{m_k V^{(k)} + e}\right)_{,\mu}\\ &= \frac{c_i V^{(i)}_{,\mu}}{m_k V^{(k)} + e} - \frac{(c_i V^{(i)} + d)m_k V^{(k)}_{,\mu}}{(m_j V^{(j)} + e)^2}\\ &= \frac{(c_i - Wm_i)V^{(i)}_{,\mu}}{m_k V^{(k)} + e}.\end{aligned}$$

Hence the relation $R^\alpha = \psi Q^\alpha$ is automatically satisfied as above stated. Specifically this means that the quantities R^α and Q^α are proportional; to insure that the factor of proportionality is positive the condition $\psi > 0$ must be imposed.

The results now obtained, combined with the first italicized result in §7, lead to the following conclusion: *All trajectories of the conservative system E will be trajectories of the conservative system D if, and only if, the quantities $h_{\mu\nu}$ and W are given by* (8.10) *and* (8.11) *where ω is given by* (8.14), *and where the constants c_i, m_i, d and e in these relations are such that $\psi > 0$ and*

(a) $$\phi^\alpha_{\mu\nu} = \delta^\alpha_\mu\phi_\nu + \delta^\alpha_\nu\phi_\mu + A_{\mu\nu}Q^\alpha,$$

(b) $$A_{\mu\nu} = \frac{1}{2}\frac{m_i g^{(i)}_{\mu\nu}}{m_k V^{(k)} + e}, \qquad m_k V^{(k)} + e > 0,$$

(c) $$A_{\mu\nu}\xi^\mu\xi^\nu \leqq 0 \text{ for all } \xi,$$

(d) $$h_{\mu\nu}\xi^\mu\xi^\nu \text{ is positive definite.}$$

9. The energy hypothesis

Suppose $s = 1$ i.e. the basis of first integrals (8.8) of the system D contains only a single member. In this case the ordinary energy integral (8.1) can be taken as a basis of these first integrals. A system D of this type will be said to satisfy the *energy hypothesis* for convenience in terminology.

Under the above hypothesis the relations (8.10), (8.11) and (8.14) become

$$h_{\mu\nu} = \omega(c - Wm)g_{\mu\nu},$$

$$W = \frac{cV + d}{mV + e},$$

$$\omega = (mV + e)^{-2/(n-1)}(c - Wm)^{-2n/(n-1)}.$$

When we substitute the above value of W in the expressions for $h_{\mu\nu}$ and ω these assume the forms

$$h_{\mu\nu} = k(mV + e)g_{\mu\nu}, \tag{9.1}$$

$$\omega = l(mV + e)^2, \tag{9.2}$$

where k and l are constants. From condition (b) in the italicized result at the end of §8 we have $mV + e > 0$; hence $k > 0$ since $g_{\mu\nu}$ and $h_{\mu\nu}$ must be the coefficients of positive definite quadratic forms. Furthermore condition (b) gives

$$A_{\mu\nu} = \frac{1}{2}\frac{mg_{\mu\nu}}{mV + e}. \tag{9.3}$$

Hence $m \leqq 0$ by condition (c). By calculation we find

$$h^{\mu\sigma} W_{,\sigma} = \left[\frac{ce - md}{k(mV + e)^3}\right] g^{\mu\sigma} V_{,\sigma}.$$

Hence,

$$\psi = \frac{ce - md}{k(mV + e)^3}.$$

Hence $\psi > 0$ if, and only if, $ce - md > 0$. There remains only condition (a) to be considered. But this condition, in which $Q^\alpha = -g^{\alpha\sigma}V_{,\sigma}$ and $4\phi = \log \omega$, is satisfied automatically on account of (9.1), (9.2) and (9.3) as can easily be seen by direct substitution.

The above constant k can be absorbed in the other constants so that we have for $h_{\mu\nu}$ and W expressions of the form

$$h_{\mu\nu} = (aV + b)g_{\mu\nu}, \qquad W = \frac{\alpha V + \beta}{aV + b}.$$

Taking account of this change of notation the italicized result at the end of §8 now leads to the following conclusion. *All trajectories of the conservative system E will be trajectories of the conservative system D, assumed to satisfy the energy hypothesis, if, and only if,*

$$(\alpha) \quad h_{\mu\nu} = (aV + b)g_{\mu\nu}, \qquad aV + b > 0,$$

$$(\beta) \quad W = \frac{\alpha V + \beta}{aV + b}, \qquad \begin{vmatrix} \alpha\beta \\ ab \end{vmatrix} > 0, \qquad a \leqq 0.$$

It is recalled that we have limited our attention to dynamical systems for which the force vectors do not vanish at points of the region under consideration. The above requirements (α) and (β) insure this condition for the system E if it holds for the system D.

10. Invariance of energy hypothesis under transformations

Starting with a conservative system[9] D, defined by the quantities $g_{\mu\nu}(x)$ and $V(x)$, let us determine quantities $h_{\mu\nu}(x)$ and $W(x)$ by one of the transformations of the group

$$h_{\mu\nu} = (aV + b)g_{\mu\nu}, \qquad aV + b \neq 0,$$

$$W = \frac{\alpha V + \beta}{aV + b}, \qquad \Delta = \begin{vmatrix} \alpha & \beta \\ a & b \end{vmatrix} \neq 0, \tag{10.1}$$

where a, b, α, β are constants. The quantities $h_{\mu\nu}$ and W then determine a conservative system E. We now prove the following result: *If the system D satisfies the energy hypothesis, the system E likewise satisfies this hypothesis.*

This result will be proved if we show that any first integral

$$\tfrac{1}{2}h^*_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + W^* = \text{const.} \tag{10.2}$$

of the system E can be expressed linearly in terms of the ordinary energy integral (8.2) of this system. In the demonstration it will be helpful to have the explicit conditions on the quantities $h^*_{\mu\nu}$ and W^* for (10.2) to be a first integral of E. From (6.3) and (6.4) it follows that these conditions are

$$W^*_{,\mu} = h^*_{\mu\nu}h^{\nu\sigma}W_{,\sigma}, \tag{10.3}$$

$$h^*_{\mu\nu;\alpha} + h^*_{\nu\alpha;\mu} + h^*_{\alpha\mu;\nu} = 0, \tag{10.4}$$

where, we recall, the "semi-colon" denotes covariant differentiation based on the Λ's. Now

$$\Lambda^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu} + \delta^\alpha_\mu\lambda_\nu + \delta^\alpha_\nu\lambda_\mu - g^{\alpha\sigma}g_{\mu\nu}\lambda_\sigma, \tag{10.5}$$

where,

$$\lambda_\mu = \frac{1}{2\sigma}\frac{\partial\sigma}{\partial x^\mu}, \qquad \sigma = aV + b.$$

Expanding the terms in (10.4) and eliminating the Λ's which thus appear by means of (10.5) it is easy to show that the resulting equations can be given the form

$$\left(\frac{h^*_{\mu\nu}}{\sigma^2} + \frac{a}{\Delta}W^*g_{\mu\nu}\right)_{,\alpha} + \left(\frac{h^*_{\nu\alpha}}{\sigma^2} + \frac{a}{\Delta}W^*g_{\nu\alpha}\right)_{,\mu} + \left(\frac{h^*_{\alpha\mu}}{\sigma^2} + \frac{a}{\Delta}W^*g_{\alpha\mu}\right)_{,\nu} = 0, \tag{10.6}$$

[9] In this section we speak of a conservative system as any system whose trajectories are determined by equations of the type (1.1) in which the Γ's are Christoffel symbols based on symmetric quantities $g_{\mu\nu}$ and $Q^\alpha = -g^{\alpha\sigma}V_{,\sigma}$. This implies that the det. $|g_{\mu\nu}|$ does not vanish but does not require the assumption that the $g_{\mu\nu}$ are the coefficients of a positive definite quadratic form.

where the covariant differentiation is now based on the Γ's. Also it is easy to show that

$$\left(\frac{\sigma W^*}{\Delta}\right)_{,\mu} = \left(\frac{h^*_{\mu\nu}}{\sigma^2} + \frac{a}{\Delta} W^* g_{\mu\nu}\right) g^{\nu\sigma} V_{,\sigma}. \tag{10.7}$$

Now put

$$g^*_{\mu\nu} = \frac{h^*_{\mu\nu}}{\sigma^2} + \frac{a}{\Delta} W^* g_{\mu\nu}, \tag{10.8}$$

$$V^* = \frac{\sigma W^*}{\Delta}. \tag{10.9}$$

Then from (10.6) and (10.7) it follows that

$$\frac{1}{2} g^*_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} + V^* = \text{const.}$$

is a first integral of the system D. But since D satisfies the energy hypothesis this means that $g^*_{\mu\nu} = c g_{\mu\nu}$ and $V^* = cV + d$ where c and d are constants. Substituting these values of $g^*_{\mu\nu}$ and V^* for the left members of (10.8) and (10.9) respectively it turns out that the resulting equations can be put in the form

$$h^*_{\mu\nu} = -\begin{vmatrix} ab \\ cd \end{vmatrix} h_{\mu\nu}, \qquad W^* = -\begin{vmatrix} ab \\ cd \end{vmatrix} + \begin{vmatrix} a\beta \\ cd \end{vmatrix}.$$

Hence the energy integral (8.2) constitutes a basis of integrals for the system E; in other words the system E satisfies the energy hypothesis as above stated.

11. Conservative systems with corresponding trajectories

On account of the result of §10 the system E in the italicized statement at the end of §9 will satisfy the energy hypothesis. Hence conditions similar to conditions (α) and (β) of this statement are necessary and sufficient for all trajectories of the system D to be trajectories of E. These conditions are of the form

$$(\gamma) \quad g_{\mu\nu} = (cW + d) h_{\mu\nu}, \qquad cW + d > 0,$$

$$(\delta) \quad V = \frac{\gamma W + \delta}{cW + d}, \qquad \begin{vmatrix} \gamma\delta \\ cd \end{vmatrix} > 0, \qquad c \leqq 0.$$

Determining the values of the above constants c, d, γ, δ in terms of the constants a, b, α, β previously used in expressing the relationship between the systems D and E, we find

$$c = \frac{-a}{\Delta}, \qquad d = \frac{\alpha}{\Delta}, \qquad \gamma = \frac{b}{\Delta}, \qquad \delta = \frac{-\beta}{\Delta}. \tag{11.1}$$

The inequality $cW + d > 0$ in (γ) and the determinant inequality in (δ) are satisfied on account of the corresponding inequalities in the conditions (α) and (β) in §9. But, since $\Delta > 0$, the inequality $c \leqq 0$ implies $-a \leqq 0$ by the first equation (11.1). Hence $a = 0$ since $a \leqq 0$ by condition (β). Hence the

relations (α) and (β) between the quantities $g_{\mu\nu}$, V and $h_{\mu\nu}$, W take the form $h_{\mu\nu} = bg_{\mu\nu}$, $b > 0$ and $W = (\alpha V + \beta)/b$, $\alpha b > 0$. Hence, *if the conservative system D satisfies the energy hypothesis, the trajectories of the conservative system E will be the same as those of D if, and only if, constants p, q and r can be found such that*

$$h_{\mu\nu} = pg_{\mu\nu}, \qquad p > 0,$$
$$W = qV + r, \qquad q > 0.$$

12. Transformation of the Hamiltonian function

Denote by H the Hamiltonian function for the conservative system D. Since the kinetic potential of D does not involve the time t explicitly the function H will not involve t explicitly and under this condition H will be equal to the sum of the kinetic and potential energies of D. Similar remarks apply to the conservative system E for which the Hamiltonian function will be denoted by L. Hence we can write

$$\frac{1}{2}g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} + V = H, \tag{12.1}$$

$$\frac{1}{2}h_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + W = L. \tag{12.2}$$

In a similar manner, using the basis of first integrals (8.8) of the system D we can define a set of *extended Hamiltonian functions* $H^{(1)}, \cdots, H^{(s)}$ with values such that

$$\frac{1}{2}g^{(i)}_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} + V^{(i)} = H^{(i)}, \qquad (i = 1, \cdots, s). \tag{12.3}$$

We shall now derive the relation between the function L of the system E and these extended functions $H^{(i)}$, associated with the system D, under the assumption that all trajectories of E are trajectories of D.

Consider the relation (12.2) along any trajectory C of the system E. Eliminate the $h_{\mu\nu}$ from (12.2) by means of (8.10) and introduce the parameter t by (4.6).[5] We thus obtain

$$L = \frac{1}{2B^2}(c_i - Wm_i)g^{(i)}_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} + W,$$
$$= \frac{(c_i - Wm_i)[H^{(i)} - V^{(i)}]}{B^2} + W,$$

when use is made of (12.3). Now (6.6) holds along C and in this relation we substitute for the quantities $1/\omega\psi$ and $A_{\mu\nu}/\omega\psi$ in accordance with (8.9) and then again apply (12.3). We thus find $B^2 = m_iH^{(i)} + e > 0$. Making this substitution for the constant B and then eliminating the quantity W by means of (8.11) we are led to the following relation

$$L = \frac{c_iH^{(i)} + d}{m_kH^{(k)} + e}. \tag{12.4}$$

Hence, *if all trajectories of the conservative system E are trajectories of the conservative system D, the Hamiltonian function L of E is related to the extended Hamiltonian functions* $H^{(1)}, \cdots, H^{(e)}$ *of the system D by the equation* (12.4) *where the conditions on the constants* c_i, m_i, d *and* e *are contained in the italicized result at the end of* §8.

From (12.4) it is seen that the relationship between L and $H^{(1)}, \cdots, H^{(e)}$ can be obtained by the substitutions $W \to L$ and $V^{(i)} \to H^{(i)}$ in the expression (8.11) for the potential of the system E. Hence, using our previous notation, it follows that *if the systems D and E are conservative and satisfy the energy hypothesis, we must have*[10]

$$(12.5) \qquad L = \frac{\alpha H + \beta}{aH + b}, \qquad aH + b > 0, \qquad \begin{vmatrix} \alpha\beta \\ ab \end{vmatrix} > 0, \qquad a \leqq 0,$$

when all trajectories of E are trajectories of D, and

$$(12.6) \qquad L = qH + r, \qquad q > 0,$$

when the trajectories of D and E are identical. The transformations of the quantities $g_{\mu\nu}$ and V corresponding to (12.5) and (12.6) are contained in the italicized results at the ends of §§9 and 11 respectively.

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[10] The condition $aH + b > 0$, proved above, implies $aV + b > 0$ on account of $a \leqq 0$ and (12.1).

MATHIEU FUNCTIONS AND THEIR CLASSIFICATION

By N. W. McLachlan

1. Introduction. In a recent paper [1], series representations of certain solutions of the following differential equations were given:

$$y'' + (a \mp 2q \cos 2z)y = 0, \qquad (1, 2)$$

$$y'' - (a \mp 2q \cosh 2z)y = 0. \qquad (3, 4)$$

(1), (2) are Mathieu equations, while (3), (4) are modified Mathieu equations derived from the former by writing iz for z. All solutions of the four equations in the form given herein, with their appropriate multipliers where necessary, are regarded as Mathieu functions. In this paper we shall: (a) extend the representations of the solutions given in reference 1, (b) define second solutions of (1)–(4), (c) introduce a number of new forms of solution of integral order, and of fractional order in the stable and unstable regions of the (a, q) plane, (d) classify the various solutions. The number of representations in the guise of series, integral relations, etc., exceeds 300. Of these about 200 have not been published hitherto. Additional representations may be derived as explained in certain sections. The results already known were taken from references 1, 1a, 2, 3, 6, 7, 9, 9a, 10. No attempt is made to show the derivation of the new formulae, as this paper would then be much too long. It is hoped that the requisite analysis will be published during the next 12 months.

Convergence of series. In the Y-, and K-Bessel series with arguments $2k \cosh z$, $2k \sinh z$, we must have $|\cosh z| > 1$, and $|\sinh z| > 1$, respectively. These series become non-uniformly convergent as $|\cosh z|$ or $|\sinh z| \to 1$. The expansions of the Y- and K-Bessel functions have logarithmic terms, and since $\sinh z$ is odd in z, if z is real, the restriction $z > 0$ is needed on $2k \sinh z$ for the function to be real.* The Mathieu *functions* represented by the Y- and K-Bessel series are continuous, since their developments in Bessel function *product* series are absolutely and *uniformly* convergent in any finite region of the z plane. Moreover, the product type series are the analytical continuations of the other series in the (finite) part of the z plane where the latter cease to apply. All series in circular and hyperbolic functions, and their p^{th} derivatives, are absolutely and *uniformly* convergent in any finite region of the z plane, so the functions they represent are continuous. For large values of z, the Bessel function product series behave computationally like asymptotic series and they are best suited for computing the functions. In integral relations with nuclei involving $Y_p(z_1)$, $K_p(z_1)$—z_1 defined below—to avoid a singularity we must have $|\cosh 2z| > 1$.

* If z is complex, $Re(z) > 0$.

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Abbreviations. It is expedient to use the following: $u_1 = 2k\cos z$, $u_2 = 2k\sin z$, $v_1 = ke^{-z}$, $v_2 = ke^{z}$, $v_1' = ke^{iz}$, $v_2' = ke^{-iz}$, $w_1 = 2k\cosh z$, $w_2 = 2k\sinh z$, $k > 0$.

$$l_{2n} = (-1)^n l'_{2n} = ce_{2n}(0, q)/A_0 \tag{5}$$

$$l_{2n+1} = (-1)^n l'_{2n+1} = ce_{2n+1}(0, q)/kA_1 \tag{6}$$

$$m_{2n+1} = (-1)^n m'_{2n+1} = se'_{2n+1}(0, q)/kB_1 \tag{7}$$

$$m_{2n+2} = (-1)^n m'_{2n+2} = se'_{2n+2}(0, q)/k^2B_2 \tag{8}$$

$$\bar{l}_{2n} = (-1)^n \bar{l}'_{2n} = ce_{2n}(\tfrac{1}{2}\pi, q)/A_0 \tag{9}$$

$$\bar{l}_{2n+1} = (-1)^n \bar{l}'_{2n+1} = -ce'_{2n+1}(\tfrac{1}{2}\pi, q)/kA_1 \tag{10}$$

$$\bar{m}_{2n+1} = (-1)^n \bar{m}'_{2n+1} = se_{2n+1}(\tfrac{1}{2}\pi, q)/kB_1 \tag{11}$$

$$\bar{m}_{2n+2} = (-1)^n \bar{m}'_{2n+2} = -se'_{2n+2}(\tfrac{1}{2}\pi, q)/k^2B_2 \tag{12}$$

$$p_{2n} = (-1)^n p'_{2n} = ce_{2n}(0, q)ce_{2n}(\tfrac{1}{2}\pi, q)/A_0 \tag{13}$$

$$p_{2n+1} = (-1)^n p'_{2n+1} = -ce_{2n+1}(0, q)ce'_{2n+1}(\tfrac{1}{2}\pi, q)/kA_1 \tag{14}$$

$$s_{2n+1} = (-1)^n s'_{2n+1} = se'_{2n+1}(0, q)se_{2n+1}(\tfrac{1}{2}\pi, q)/kB_1 \tag{15}$$

$$s_{2n+2} = (-1)^{n+1} s'_{2n+2} = se'_{2n+2}(0, q)se'_{2n+2}(\tfrac{1}{2}\pi, q)/k^2B_2 \tag{16}$$

$\sum$ means $\sum_{r=0}^{\infty}$, $\overline{\sum}$ means $\sum_{r=0}^{\infty}(-1)^r$; *Re* means the 'real' part, and *Im* the 'imaginary' part; $m, n = 0, 1, 2, \cdots$, positive integers. a, q are real unless stated otherwise; $k = +q^{\frac{1}{4}} = \frac{1}{2}k_1h$; h is the semi-interfocal distance for the fundamental ellipse; $z_1 = k[2(\cosh 2z + \cos 2u)]^{\frac{1}{2}} = k_1(x^2 + y^2)^{\frac{1}{2}} = k_1 r$, r being the radius vector of the point z, u. In elliptical coordinates $x = h\cosh z\cos u$, $y = h\sinh z\sin u$, x and y being the cartesian coordinates of a point P, on a confocal ellipse, distant r from the centre. $y/x = \tan\alpha = \tanh z\tan u$.

In formulae on the right hand side of the $=$ sign in succeeding sections, and in (1)–(4) above, k, q are positive, unless the contrary is indicated. k, q are positive in the multipliers (5)–(16).

The characteristic numbers are indicated in parentheses on the right side of the page, e.g. (a_{2n}), (b_{2n+1}), $(a_{2n+\beta})$, $(b_{2n+\mu})$ reference being to that for $q > 0$.

2. $ce_m(z, q)$, $se_m(z, q)$, **first (periodic) solutions of** (*1*) §*1*.

$$ce_{2n}(z, q) = \Sigma A_{2r}\cos 2rz \qquad (a_{2n}) \tag{1}$$

$$= l_{2n}\bar{\Sigma}A_{2r}J_{2r}(u_1) \tag{2}$$

$$= l_{2n}\bar{\Sigma}A_{2r}I_{2r}(u_2) \tag{3}$$

$$= (p_{2n}/A_0)\bar{\Sigma}A_{2r}J_r(v_1')J_r(v_2') \tag{4}$$

$$= (l_{2n}/\pi)\int_0^{\pi}\cos(u_1\cos u)\,ce_{2n}(u, q)du \tag{5}$$

$$= (l_{2n}/\pi)\int_0^\pi \cosh\,(u_2 \sin u)\, ce_{2n}(u, q)\, du \tag{6}$$

$$ce_{2n+1}(z, q) = \Sigma A_{2r+1} \cos\,(2r+1)z \qquad (a_{2n+1}) \tag{7}$$

$$= l_{2n+1}\bar{\Sigma}A_{2r+1}J_{2r+1}(u_1) \tag{8}$$

$$= l_{2n+1} \cot z\, \bar{\Sigma}(2r+1)A_{2r+1}I_{2r+1}(u_2) \tag{9}$$

$$= (p_{2n+1}/A_1)\bar{\Sigma}A_{2r+1}[J_r(v_1')J_{r+1}(v_2') + J_{r+1}(v_1')J_r(v_2')] \tag{10}$$

$$= 2(p_{2n+1}/A_1)\,\bar{\Sigma}A_{2r+1}\,Re\{J_r(v_1')\,J_{r+1}(v_2')\}, \qquad z \text{ real} \tag{11}$$

$$= (l_{2n+1}/\pi)\int_0^\pi \sin(u_1 \cos u) ce_{2n+1}(u, q)du \tag{12}$$

$$= (l_{2n+1}/\pi)\; u_1\int_0^\pi \cosh\,(u_2 \sin u)\, \cos u\, ce_{2n+1}\,(z, q)\, du \tag{13}$$

$$= -(l_{2n+1}/\pi)\, \cot z\int_0^\pi \sinh\,(u_2 \sin u)\, ce'_{2n+1}(u, q)du \tag{14}$$

$$se_{2n+1}(z, q) = \Sigma B_{2r+1} \sin\,(2r+1)z \qquad (b_{2n+1}) \tag{15}$$

$$= \bar{m}_{2n+1} \tan z\, \bar{\Sigma}(2r+1)B_{2r+1}J_{2r+1}(u_1) \tag{16}$$

$$= m_{2n+1}\bar{\Sigma}B_{2r+1}\, I_{2r+1}(u_2) \tag{17}$$

$$= i^{-1}(s_{2n+1}/B_1)\bar{\Sigma}B_{2r+1}[J_r(v_1')J_{r+1}(v_2') - J_{r+1}(v_1')J_r(v_2')] \tag{18}$$

$$= 2(s_{2n+1}/B_1)\bar{\Sigma}B_{2r+1}Im\{J_r(v_1')J_{r+1}(v_2')\}, \qquad z \text{ real} \tag{19}$$

$$= (m_{2n+1}/\pi)\int_0^\pi \sinh\,(u_2 \sin u) se_{2n+1}(u, q)du \tag{20}$$

$$= (\bar{m}_{2n+1}/\pi)u_2\int_0^\pi \cos\,(u_1 \cos u)\, \sin u\, se_{2n+1}(u, q)du \tag{21}$$

$$= (\bar{m}_{2n+1}/\pi)\, \tan z\int_0^\pi \sin\,(u_1 \cos u)\, se'_{2n+1}(u, q)du \tag{22}$$

$$se_{2n+2}(z, q) = \Sigma B_{2r+2} \sin\,(2r+2)z \qquad (b_{2n+2}) \tag{23}$$

$$= \bar{m}_{2n+2} \tan z\, \bar{\Sigma}(2r+2)B_{2r+2}J_{2r+2}(u_1) \tag{24}$$

$$= m_{2n+2} \cot z\, \bar{\Sigma}(2r+2)B_{2r+2}I_{2r+2}(u_2) \tag{25}$$

$$= i(S_{2n+2}/B_2)\bar{\Sigma}B_{2r+2}[J_r(v_1')J_{r+2}(v_2') - J_{r+2}(v_1')J_r(v_2')] \tag{26}$$

$$= -2(S_{2n+2}/B_2)\bar{\Sigma}B_{2r+2}Im\{J_r(v_1')J_{r+2}(v_2')\}, \qquad z \text{ real} \tag{27}$$

$$= (m_{2n+2}/\pi)u_1\int_0^\pi \sinh\,(u_2 \sin u)\, \cos u\, se_{2n+2}(u, q)du \tag{28}$$

$$= (\bar{m}_{2n+2}/\pi)u_2\int_0^\pi \sin\,(u_1 \cos u)\, \sin u\, se_{2n+2}(u, q)du \tag{29}$$

See Corrections
V. 26 - no. 1, 1947. p. 78

$$= -(m_{2n+2}/\pi)\cot z\int_0^\pi \cosh(u_2\sin u)\,se'_{2n+2}(u,q)\,du \tag{30}$$

$$= -(\bar{m}_{2n+2}/\pi)\tan z\int_0^\pi \cos(u_1\cos u)\,se'_{2n+2}(u,q)\,du \tag{31}$$

If z is real, the functions of even order have period π, while those of odd order have period 2π. As $q\to 0$, $A_m^{(m)}$, $B_m^{(m)}\to 1$, but the other A, $B\to 0$. Hence as $q\to 0$, $ce_m(z,q)\to +\cos mz$, and $se_m(z,q)\to +\sin mz$, these being solutions of (1) §1 with $q=0$, $a=m^2$. The positive sign is conventional, while the superscript (m) indicates the order of the function.

Additional developments and integral relations may be derived from (4), (5), (7), etc., §6, by applying the relations between ce_m and Ce_m, se_m and Se_m which are given in §6.

3. $fe_m(z,q)$, $ge_m(z,q)$, second (non-periodic) solutions of (1) §1.

$$fe_{2n}(z,q) = C_{2n}(q)[z\,ce_{2n}(z,q) + \Sigma f_{2r+2}^{(2n)}\sin(2r+2)z] \qquad (a_{2n}) \tag{1}$$

$$= C'_{2n}(p_{2n}/A_0)\bar{\Sigma}A_{2r}\,Im\{J_r(v'_1)Y_r(v'_2)\}, \qquad z \text{ real} \tag{2}$$

$$fe_{2n+1}(z,q) = C_{2n+1}(q)[z\,ce_{2n+1}(z,q) + \Sigma f_{2r+1}^{(2n+1)}\sin(2r+1)z] \qquad (a_{2n+1}) \tag{3}$$

$$= C'_{2n+1}(p_{2n+1}/A_1)\bar{\Sigma}A_{2r+1}Im\{J_r(v'_1)Y_{r+1}(v'_2) + J_{r+1}(v'_1)Y_r(v'_2)\}, \qquad z \text{ real} \tag{4}$$

$$ge_{2n+1}(z,q) = S_{2n+1}(q)[z\,se_{2n+1}(z,q) + \Sigma g_{2r+1}^{(2n+1)}\cos(2r+1)z] \qquad (b_{2n+1}) \tag{5}$$

$$= S'_{2n+1}(s_{2n+1}/B_1)\bar{\Sigma}B_{2r+1}Re\{J_r(v'_1)Y_{r+1}(v'_2) - J_{r+1}(v'_1)Y_r(v'_2)\}, \qquad z \text{ real} \tag{6}$$

$$ge_{2n+2}(z,q) = S_{2n+2}(q)[z\,se_{2n+2}(z,q) + \Sigma g_{2r}^{(2n+2)}\cos 2rz] \qquad (b_{2n+2}) \tag{7}$$

$$= S'_{2n+2}(s_{2n+2}/B_2)\bar{\Sigma}B_{2r+2}Re\{J_r(v'_1)Y_{r+2}(v'_2) - J_{r+2}(v'_1)Y_r(v'_2)\}, \qquad z \text{ real} \tag{8}$$

Normalisation. W. G. Bickley has suggested the rule that

$$C_m^2(q)\sum f_p^2 = S_{2n+1}^2(q)\sum g_{2r+1}^2 = S_{2n+2}^2(q)\left[2g_0^2 + \sum_{r=1}^{\infty} g_{2r}^2\right] = 1. \tag{9}$$

The f, g may be computed in terms of the known parameters a, q and the tabulated coefficients $A_p^{(m)}$, $B_p^{(m)}$[6]. Then we obtain

$$C_m(q) = 1/[\Sigma f_p^2]^{\frac{1}{2}}, \tag{10}$$

$$S_{2n+1}(q) = 1/[\Sigma g_{2r+1}^2]^{\frac{1}{2}}, \tag{11}$$

and

$$S_{2n+2}(q) = 1 \Big/ \left[2g_0^2 + \sum_{r=1}^{\infty} g_{2r}^2 \right]^{\frac{1}{2}}. \tag{12}$$

C_m, S_m vary with q. As $q \to 0$, $C_0(q) \to +1$, this being conventional; $C_1(q)$ and $S_1(q) \to q$; $C_2(q)$ and $S_2(q) \to \frac{1}{2}q^2$; $C_m(q)$ and $S_m(q) \to q^m/2^{2m-2}(m-1)!\,m!$ Also $C_m(q)f_p^{(m)}$ and $S_m(q)g_p^{(m)} \to 0$ as $q \to 0$, unless $p = m$, when they tend to $+1$. Since $C_m(q)$ and $S_m(q) \to 0$ when $q \to 0$ and $m > 0$, it follows that $f_m^{(m)} \to 1/C_m(q) \to +\infty$, $g_m^{(m)} \to 1/S_m(q) \to +\infty$, but this is inconsequential. Hence as $q \to 0$

$$fe_m(z, q) \to \sin mz, \quad \text{and} \quad ge_m(z, q) \to \cos mz, \tag{13}$$

these being solutions of (1) §1 with $q = 0$, $a = m^2$.

Having normalised the above functions, C_m', S_m' may be calculated by aid of the relations (see also (8), (9) in the Appendix):

$$C_{2n}' = -\tfrac{1}{2}\pi fe_{2n}'(0, q)/p_{2n}l_{2n} \tag{14}$$

$$C_{2n+1}' = -\tfrac{1}{2}\pi fe_{2n+1}'(0, q)/p_{2n+1}l_{2n+1} \tag{15}$$

$$S_{2n+1}' = -\tfrac{1}{2}\pi ge_{2n+1}(0, q)/s_{2n+1}\bar{m}_{2n+1} \tag{16}$$

$$S_{2n+2}' = \tfrac{1}{2}\pi ge_{2n+2}(0, q)/s_{2n+2}\bar{m}_{2n+2}. \tag{17}$$

Excepting $fe_0(z, q)$, the above functions alternate* with linearly increasing amplitude when z (large) increases; and they tend to $\pm\infty$ as $z \to +\infty$. In the range $0 \leqq q < \infty$, $ce_0(z, q) > 0$. Thus in (1) as $z \to +\infty$, $\Sigma f_{2r+2} \sin(2r+2)z$ is negligible compared with $zce_0(z, q)$, so $fe_0(z, q) \to +\infty$ with z. Its graph for z large is the line $y = C_0(q)z$, with a peturbation due to $ce_0(z, q)$ superimposed.

4. $ce_m(z, -q)$, $se_m(z, -q)$, first (periodic) solutions of (2) §1.

$$ce_{2n}(z, -q) = (-1)^n ce_{2n}(\tfrac{1}{2}\pi - z, q) = (-1)^n \Sigma A_{2r} \cos 2rz \qquad (a_{2n}) \tag{1}$$

$$= l_{2n}' \Sigma A_{2r} J_{2r}(u_2) \tag{2}$$

$$= l_{2n}' \Sigma A_{2r} I_{2r}(u_1) \tag{3}$$

$$= (p_{2n}'/A_0) \Sigma A_{2r} I_r(v_1') I_r(v_2') \tag{4}$$

$$ce_{2n+1}(z, -q) = (-1)^n se_{2n+1}(\tfrac{1}{2}\pi - z, q) = (-1)^n \Sigma B_{2r+1} \cos(2r+1)z \qquad (b_{2n+1}) \tag{5}$$

$$= \bar{m}_{2n+1}' \cot z \, \Sigma(2r+1) B_{2r+1} J_{2r+1}(u_2) \tag{6}$$

$$= m_{2n+1}' \Sigma B_{2r+1} I_{2r+1}(u_1) \tag{7}$$

$$= -(s_{2n+1}'/B_1) \Sigma B_{2r+1} [I_r(v_1') I_{r+1}(v_2') + I_{r+1}(v_1') I_r(v_2')] \tag{8}$$

$$= -2(s_{2n+1}'/B_1) \Sigma B_{2r+1} Re\{I_r(v_1') I_{r+1}(v_2')\}, \qquad z \text{ real} \tag{9}$$

* It is convenient to regard an alternating function as one which changes sign repeatedly as z (real) increases. The function may be continuous, finitely discontinuous (piecewise continuous), periodic or otherwise, while its amplitude may vary boundedly.

$$se_{2n+1}(z, -q) = (-1)^n ce_{2n+1}(\tfrac{1}{2}\pi - z, q) = (-1)^n \bar{\Sigma} A_{2r+1} \sin(2r+1)z \qquad (a_{2n+1}) \quad (10)$$

$$= l'_{2n+1} \bar{\Sigma} A_{2r+1} J_{2r+1}(u_2) \quad (11)$$

$$= l'_{2n+1} \tan z \, \bar{\Sigma}(2r+1) A_{2r+1} I_{2r+1}(u_1) \quad (12)$$

$$= i(p'_{2n+1}/A_1) \bar{\Sigma} A_{2r+1} [I_r(v'_1) I_{r+1}(v'_2) - I_{r+1}(v'_1) I_r(v'_2)] \quad (13)$$

$$= -2(p'_{2n+1}/A_1) \bar{\Sigma} A_{2r+1} Im\{I_r(v'_1) I_{r+1}(v'_2)\}, \qquad z \text{ real} \quad (14)$$

$$se_{2n+2}(z, -q) = (-1)^n se_{2n+2}(\tfrac{1}{2}\pi - z, q) = (-1)^n \bar{\Sigma} B_{2r+2} \sin(2r+2)z \qquad (b_{2n+2}) \quad (15)$$

$$= m'_{2n+2} \cot z \, \bar{\Sigma}(2r+2) B_{2r+2} J_{2r+2}(u_2) \quad (16)$$

$$= m'_{2n+2} \tan z \, \bar{\Sigma}(2r+2) B_{2r+2} I_{2r+2}(u_1) \quad (17)$$

$$= i^{-1}(s'_{2n+2}/B_2) \bar{\Sigma} B_{2r+2} [I_r(v'_1) I_{r+2}(v'_2) - I_{r+2}(v'_1) I_r(v'_2)] \quad (18)$$

$$= 2(s'_{2n+2}/B_2) \bar{\Sigma} B_{2r+2} Im\{I_r(v'_1) I_{r+2}(v'_2)\}, \; z \text{ real}. \quad (19)$$

If z is real, the functions of $\begin{Bmatrix}\text{even}\\ \text{odd}\end{Bmatrix}$ order have period $\begin{Bmatrix}\pi\\ 2\pi\end{Bmatrix}$. When $q \to 0$ the behaviour of A, B stated after (31) § 2, is such that $se_m(z, -q) \to +\cos mz$, and $se_m(z, -q \to +\sin mz$. The multiplier $(-1)^n$ ensures the conventional $+$ sign. Additional representations may be derived from §2, by applying (1), (5), (10), (15) above.

5. $fe_m(z, -q)$, $ge_m(z, -q)$, **second (non-periodic) solutions of** (2) §1.

$$fe_{2n}(z, -q) = (-1)^{n+1} fe_{2n}(\tfrac{1}{2}\pi - z, q) \qquad (a_{2n}) \quad (1)$$

$$= -C_{2n}(q)[(\tfrac{1}{2}\pi - z) ce_{2n}(z, -q) + (-1)^n \bar{\Sigma} f_{2r+2} \sin(2r+2)z] \quad (2)$$

$$= C'_{2n}[ce_{2n}(z, -q) - 2 fek_{2n}(z, -q)], \qquad z \text{ real} \quad (3)$$

$$fe_{2n+1}(z, -q) = (-1)^n ge_{2n+1}(\tfrac{1}{2}\pi - z, q) \qquad (b_{2n+1}) \quad (4)$$

$$= S_{2n+1}(q)[(\tfrac{1}{2}\pi - z) ce_{2n+1}(z, -q) + (-1)^n \bar{\Sigma} g_{2r+1} \sin(2r+1)z] \quad (5)$$

$$= S'_{2n+1}[ce_{2n+1}(z, -q) - 2 fek_{2n+1}(z, -q)], \qquad z \text{ real} \quad (6)$$

$$ge_{2n+1}(z, -q) = (-1)^n fe_{2n+1}(\tfrac{1}{2}\pi - z, q) \qquad (a_{2n+1}) \quad (7)$$

$$= C_{2n+1}(q)[(\tfrac{1}{2}\pi - z) se_{2n+1}(z, -q) + (-1)^n \bar{\Sigma} f_{2r+1} \cos(2r+1)z] \quad (8)$$

$$= C'_{2n+1}[se_{2n+1}(z, -q) - 2 gek_{2n+1}(z, -q)], \qquad z \text{ real} \quad (9)$$

$$ge_{2n+2}(z, -q) = (-1)^n ge_{2n+2}(\tfrac{1}{2}\pi - z, q) \qquad (b_{2n+2}) \quad (10)$$

$$= S_{2n+2}(q)[(\tfrac{1}{2}\pi - z)se_{2n+2}(z, -q) + (-1)^n\bar{\Sigma}\, g_{2r}\cos 2rz \qquad (11)$$

$$= S'_{2n+2}[se_{2n+2}(z, -q) - 2gek_{2n+2}(z, -q)], \qquad z \text{ real.} \qquad (12)$$

For z real and $q > 0$,

$$fek_{2n}(z, -q) = (p'_{2n}/\pi A_0)\bar{\Sigma}A_{2r}Im\{I_r(v'_1)K_r(v'_2)\} \qquad (13)$$

$$fek_{2n+1}(z, -q) = (s'_{2n+1}/\pi B_1)\Sigma B_{2r+1}Im\{I_r(v'_1)K_{r+1}(v'_2) - I_{r+1}(v'_1)K_r(v'_2)\} \qquad (14)$$

$$gek_{2n+1}(z, -q) = (p'_{2n+1}/\pi A_1)\Sigma A_{2r+1}Re\{I_r(v'_1)K_{r+1}(v'_2) + I_{r+1}(v'_1)K_r(v'_2)\} \qquad (15)$$

$$gek_{2n+2}(z, -q) = (s'_{2n+2}/\pi B_2)\Sigma B_{2r+2}Re\{I_r(v'_1)K_{r+2}(v'_2) - I_{r+2}(v'_1)K_r(v'_2)\} \qquad (16)$$

As $z \to +\infty$ the behaviour of (2), (5), (8), (11) is similar to that of the corresponding functions in §3. When $q \to 0$, $fe_m(z, -q) \to +\sin mz$, and $ge_m(z, -q) \to +\cos mz$, these being solutions of (2) § 1, with $q = 0$, $a = m^2$. The multipliers $(-1)^{n+1}$ ensure the conventional positive signs.

6. $Ce_m(z, q)$, $Se_m(z, q)$ **first (periodic) solutions of** (3) §1.

$$Ce_{2n}(z, q) = ce_{2n}(iz, q) = \Sigma A_{2r}\cosh 2rz \qquad (a_{2n}) \qquad (1)$$

$$= l_{2n}\bar{\Sigma}A_{2r}J_{2r}(w_1) \qquad (2)$$

$$= l_{2n}\Sigma A_{2r}J_{2r}(w_2) \qquad (3)$$

$$= [p_{2n}/ce_{2n}(u, q)]\bar{\Sigma}A_{2r}\cos 2r\alpha J_{2r}(z_1) \qquad (4)$$

$$= (p_{2n}/\pi A_0)\int_0^{2\pi} J_0(z_1)ce_{2n}(u, q)\,du \qquad (5)$$

$$= (p_{2n}/A_0)\bar{\Sigma}A_{2r}J_r(v_1)J_r(v_2) \qquad (6)$$

$$= (2l_{2n}/\pi)\int_0^{\infty} \sin(w_1\cosh u)Ce_{2n}(u, q)\,du \qquad (7)$$

$$Ce_{2n+1}(z, q) = ce_{2n+1}(iz, q) = \Sigma A_{2r+1}\cosh(2r+1)z \qquad (a_{2n+1}) \qquad (8)$$

$$= l_{2n+1}\bar{\Sigma}A_{2r+1}J_{2r+1}(w_1) \qquad (9)$$

$$= l_{2n+1}\coth z\,\Sigma(2r+1)A_{2r+1}J_{2r+1}(w_2) \qquad (10)$$

$$= [p_{2n+1}/ce_{2n+1}(u, q)]\bar{\Sigma}A_{2r+1}\cos(2r+1)\alpha J_{2r+1}(z_1) \qquad (11)$$

$$= (p_{2n+1}/\pi A_1)\int_0^{2\pi} J_1(z_1)\cos\alpha\, ce_{2n+1}(u, q)\,du \qquad (12)$$

$$= (p_{2n+1}/A_1)\bar{\Sigma}A_{2r+1}[J_r(v_1)J_{r+1}(v_2) + J_{r+1}(v_1)J_r(v_2)] \qquad (13)$$

$$= -(2l_{2n}/\pi)\int_0^{\infty} \cos(w_1\cosh u)Ce_{2n+1}(u, q)\,du \qquad (14)$$

$$Se_{2n+1}(z, q) = i^{-1}se_{2n+1}(iz, q) = \Sigma B_{2r+1}\sinh(2r+1)z \qquad (b_{2n+1}) \qquad (15)$$

$$= \bar{m}_{2n+1} \tanh z \, \bar{\Sigma}(2r+1) B_{2r+1} J_{2r+1}(w_1) \tag{16}$$

$$= m_{2n+1} \Sigma B_{2r+1} J_{2r+1}(w_2) \tag{17}$$

$$= [se_{2n+1}/se'_{2n+1}(u, q)] \bar{\Sigma} B_{2r+1} \sin (2r+1)\alpha J_{2r+1}(z_1) \tag{18}$$

$$= (\varepsilon_{2n+1}/\pi B_1) \int_0^{2\pi} J_1(z_1) \sin \alpha \, se_{2n+1}(u, q) \, du \tag{19}$$

$$= (\varepsilon_{2n+1}/B_1) \bar{\Sigma} B_{2r+1} [J_r(v_1) J_{r+1}(v_2) - J_{r+1}(v_1) J_r(v_2)] \tag{20}$$

$$= -(2\bar{m}_{2n+1}/\pi) w_2 \int_0^{\infty} \sin (w_1 \cosh u) \sinh u \, Se_{2n+1}(u, q) \, du \tag{21}$$

$$= -(2\bar{m}_{2n+1}/\pi) \tanh z \int_0^{\infty} \cos (w_1 \cosh u) \, Se'_{2n+1}(u, q) \, du \tag{22}$$

$$Se_{2n+2}(z, q) = i^{-1} se_{2n+2}(iz, q) = \Sigma B_{2r+2} \sinh (2r+2)z \qquad (b_{2n+2}) \tag{23}$$

$$= \bar{m}_{2n+2} \tanh z \, \bar{\Sigma}(2r+2) B_{2r+2} J_{2r+2}(w_1) \tag{24}$$

$$= m_{2n+2} \coth z \, \bar{\Sigma}(2r+2) B_{2r+2} J_{2r+2}(w_2) \tag{25}$$

$$= -[\varepsilon_{2n+2}/se'_{2n+2}(u, q)] \bar{\Sigma} B_{2r+2} \sin (2r+2) z J_{2r+2}(z_1) \tag{26}$$

$$= (\varepsilon_{2n+2}/\pi B_2) \int_0^{2\pi} J_2(z_1) \sin 2\alpha \, se_{2n+2}(u, q) \, du \tag{27}$$

$$= -(\varepsilon_{2n+2}/B_2) \bar{\Sigma} B_{2r+2} [J_r(v_1) J_{r+2}(v_2) - J_{r+2}(v_1) J_r(v_2)] \tag{28}$$

$$= (2\bar{m}_{2n+2}/\pi) w_2 \int_0^{\infty} \cos (w_1 \cosh u) \sinh u \, Se_{2n+2}(u, q) \, du \tag{29}$$

$$= -(2\bar{m}_{2n+2}/\pi) \tanh z \int_0^{\infty} \sin (w_1 \cosh u) \, Se'_{2n+2}(u, q) \, du \tag{30}$$

If z is real, these functions have period πi, or $2\pi i$, according as the order is even or odd. As $q \to 0$, $Ce_m(z, q) \to +\cosh mz$, and $Se_m(z, q) \to +\sinh mz$, these being solutions of (3) §1, with $q = 0$, $a = m^2$.

Additional integral relations may be obtained from (5), (6), etc. in §2, by applying (1), (8), (15), (23) above. In (7), (14), (21), (22), (29), (30), k, z, real > 0.

7. $Fe_m(z, q)$, $Ge_m(z, q)$, **alternative second (non-periodic) solutions of** (3) 1§.

$$Fe_{2n}(z, q) = i^{-1} fe_{2n}(iz, q) = C_{2n}(q)[z \, Ce_{2n}(z, q) + \Sigma f^{(2n)}_{2r+2} \sinh (2r+2)z] \qquad (a_{2n}) \tag{1}$$

$$Fe_{2n+1}(z, q) = i^{-1} fe_{2n+1}(iz, q) = C_{2n+1}(q)[z \, Ce_{2n+1}(z, q) + \Sigma f^{(2n+1)}_{2r+1} \sinh (2r+1)z] \qquad (a_{2n+1}) \tag{2}$$

$$Ge_{2n+1}(z, q) = ge_{2n+1}(iz, q) = -S_{2n+1}(q)[z \, Se_{2n+1}(z, q) - \Sigma g^{(2n+1)}_{2r+1} \cosh (2r+1)z] \qquad (b_{2n+1}) \tag{3}$$

$$Ge_{2n+2}(z, q) = ge_{2n+2}(iz, q) = -S_{2n+2}(q)[z\, Se_{2n+2}(z, q) - \Sigma g_{2r}^{(2n+2)} \cosh 2rz] \qquad (b_{2n+2}) \quad (4)$$

By virtue of the relationships in § 9, as $z \to +\infty$ the behaviour of these functions may be inferred from that of $Ce_m(z, q)$, $Fey_m(z, q)$, etc, which is indicated in §28.

As $q \to 0$ it may be deduced from § 3 that $Fe_m(z, q) \to +\sinh mz$, and $Ge_m(z, q) \to +\cosh mz$, these being solutions of (3)§1 when $q = 0$, $a = m^2$.

8. $Fey_m(z, q)$, $Gey_m(z, q)$, second (non-periodic) solutions of (3)§1.

$$Fey_{2n}(z, q) = l_{2n}\bar{\Sigma}A_{2r}Y_{2r}(w_1) \qquad (a_{2n}) \quad (1)$$

$$= l_{2n}\Sigma A_{2r}Y_{2r}(w_2) \quad (2)$$

$$= (p_{2n}/\pi A_0)\int_0^{2\pi} Y_0(z_1)ce_{2n}(u, q)\, du \quad (3)$$

$$= (p_{2n}/A_0)\bar{\Sigma}A_{2r}J_r(v_1)Y_r(v_2) \quad (4)$$

$$= -(2l_{2n}/\pi)\int_0^{\infty} \cos(w_1 \cosh u)\, Ce_{2n}(u, q)\, du \quad (5)$$

$$Fey_{2n+1}(z, q) = l_{2n+1}\bar{\Sigma}A_{2r+1}Y_{2r+1}(w_1) \qquad (a_{2n+1}) \quad (6)$$

$$= l_{2n+1}\coth z\, \Sigma(2r+1)A_{2r+1}Y_{2r+1}(w_2) \quad (7)$$

$$= (p_{2n+1}/\pi A_1)\int_0^{2\pi} Y_1(z_1)\cos\alpha\, ce_{2n+1}(u, q)\, du \quad (8)$$

$$= (p_{2n+1}/A_1)\bar{\Sigma}A_{2r+1}[J_r(v_1)Y_{r+1}(v_2) + J_{r+1}(v_1)Y_r(v_2)] \quad (9)$$

$$= -(2l_{2n+1}/\pi)\int_0^{\infty} \sin(w_1 \cosh u)\, Ce_{2n+1}(u, q)\, du \quad (10)$$

$$Gey_{2n+1}(z, q) = \bar{m}_{2n+1}\tanh z\, \bar{\Sigma}(2r+1)B_{2r+1}Y_{2r+1}(w_1) \qquad (b_{2n+1}) \quad (11)$$

$$= m_{2n+1}\Sigma B_{2r+1}Y_{2r+1}(w_2) \quad (12)$$

$$= (s_{2n+1}/\pi B_1)\int_0^{2\pi} Y_1(z_1)\sin\alpha\, se_{2n+1}(u, q)\, du \quad (13)$$

$$= (s_{2n+1}/B_1)\bar{\Sigma}B_{2r+1}[J_r(v_1)Y_{r+1}(v_2) - J_{r+1}(v_1)Y_r(v_2)] \quad (14)$$

$$= (2\bar{m}_{2n+1}/\pi)w_2\int_0^{\infty} \cos(w_1 \cosh u)\sinh u\, Se_{2n+1}(u, q)\, du \quad (15)$$

$$= -(2\bar{m}_{2n+1}/\pi)\tanh z\int_0^{\infty} \sin(w_1 \cosh u)\, Se'_{2n+1}(u, q)\, du \quad (16)$$

$$Gey_{2n+2}(z, q) = \bar{m}_{2n+2}\tanh z\, \bar{\Sigma}(2r+2)B_{2r+2}Y_{2r+2}(w_1) \qquad (b_{2n+2}) \quad (17)$$

$$= m_{2n+2}\coth z\, \Sigma(2r+2)B_{2r+2}Y_{2r+2}(w_2) \quad (18)$$

$$= (s_{2n+2}/\pi B_2) \int_0^{2\pi} Y_2(z_1) \sin 2\alpha \; se_{2n+2}(u, q)\, du \tag{19}$$

$$= -(s_{2n+2}/B_2) \Sigma B_{2r+2}[J_r(v_1) Y_{r+2}(v_2) - J_{r+2}(v_1) Y_r(v_2)] \tag{20}$$

$$= (2\bar{m}_{2n+2}/\pi) w_2 \int_0^{\infty} \sin(w_1 \cosh u) \sinh u \; Se_{2n+2}(u, q)\, du \tag{21}$$

$$= (2\bar{m}_{2n+2}/\pi) \tanh z \int_0^{\infty} \cos(w_1 \cosh u) \; Se'_{2n+2}(u, q)\, du. \tag{22}$$

These functions are non-periodic by virtue of the logarithmic term in the expansion of the Y-Bessel function. In (5), (10), (15), (16), (21), (22), $\cosh z > 1$ in w_1, $\sinh z > 0$ in w_2; k, z, real > 0.

9. Relationship between $Fey_m(z, q)$, $Fe_m(z, q)$; $Gey_m(z, q)$, $Ge_m(z, q)$.

Fey_m and Fe_m are second solutions of (3)§1, which correspond to the first solution Ce_m. Since the equation is linear of the second order we may write,

$$Fey_m(z, q = ACe_m(z, q) + BFe_m(z, q), \tag{1}$$

where A, B are constants. Now $Fe_m(0, q) = Ce'_m(0, q) = 0$, but neither $Fey_m(0, q)$ nor $Fey'_m(0, q)$ are zero, so we find that

$$Fey_m(z, q) = \frac{Fey_m(0, q)}{Ce_m(0, q)} Ce_m(z, q) + \frac{Fey'_m(0, q)}{Fe'_m(0, q)} Fe_m(z, q). \tag{2}$$

Similarly we get

$$Gey_m(z, q) = \frac{Gey'_m(0, q)}{Se'_m(0, q)} Se_m(z, q) + \frac{Gey_m(0, q)}{Ge_m(0, q)} Ge_m(z, q). \tag{3}$$

Note that $Ce_m(0, q) = ce_m(0, q)$, and $Se'_m(0, q) = se'_m(0, q)$. The following results were deduced by aid of the expansions in Bessel function products:

$$Fey'_{2n}(0, q) = (2/\pi) p_{2n} l_{2n} \tag{4}$$

$$Fey'_{2n+1}(0, q) = (2/\pi) p_{2n+1} l_{2n+1}) \tag{5}$$

$$Gey_{2n+1}(0, q) = -(2/\pi) s_{n+1} \bar{m}_{2n+1} \tag{6}$$

$$Gey_{2n+2}(0, q) = (2/\pi) s_{2n+2} \bar{m}_{2n+2}. \tag{7}$$

$Fey_m(0, q)$ and $Gey'_m(0, q)$ may be obtained by aid of the B.F. product representations.

The relationships for equation (4)§1 are the same as those at (2), (3) with $-q$ for q, but (4)–(7) are then inapplicable.

10. $Fek_m(z, q)$, $Gek_m(z, q)$, second (non-periodic) solutions of (3)§1.

$$Fek_{2n}(z, q) = (l_{2n}/\pi) \Sigma A_{2r} K_{2r}(-iw_1) \qquad (a_{2n}) \tag{1}$$

$$= (l_{2n}/\pi) \bar{\Sigma} A_{2r} K_{2r}(-iw_2). \tag{2}$$

$$= i(p_{2n}/2\pi A_0) \int_0^{2\pi} H_0^{(1)}(z_1) ce_{2n}(u, q)\, du \tag{3}$$

$$= (p_{2n}/\pi A_0)\ \Sigma A_{2r} I_r(iw_1) K_r(-iw_2) \tag{4}$$

$$= (l_{2n}/\pi) \int_0^{\infty} e^{iw_1 \cosh u} Ce_{2n}(u, q)\, du \tag{5}$$

$$= \tfrac{1}{2} i\, Me_{2n}^{(1)}(z, q). \tag{6}$$

$$Fek_{2n+1}(z, q) = (l_{2n+1}/\pi)\Sigma A_{2r+1} K_{2r+1}(-iw_1) \qquad (a_{2n+1}) \tag{7}$$

$$= (l_{2n+1}/\pi)\ \coth z\ \bar{\Sigma}(2r+1) A_{2r+1} K_{2r+1}(-iw_2) \tag{8}$$

$$= -(p_{2n+1}/2\pi A_1) \int_0^{2\pi} H_1^{(1)}(z_1) \cos\alpha\ Ce_{2n+1}(u, q)\, du \tag{9}$$

$$= (p_{2n+1}/\pi A_1)\Sigma A_{2r+1}[I_r(iw_1) K_{r+1}(-iw_2) + I_{r+1}(iw_1) K_r(-iw_2)] \tag{10}$$

$$= (l_{2n+1}/\pi) \int_0^{\infty} e^{iw_1 \cosh u} Ce_{2n+1}(u, q)\, du \tag{11}$$

$$= -\tfrac{1}{2}\, Me_{2n+1}^{(1)}(z, q) \tag{12}$$

$$Gek_{2n+1}(z, q) = (\bar{m}_{2n+1}/\pi)\ \tanh z\ \Sigma(2r+1) B_{2r+1} K_{2r+1}(-iw_1) \qquad (b_{2n+1}) \tag{13}$$

$$= (m_{2n+1}/\pi)\bar{\Sigma} B_{2r+1} K_{2r+1}(-iw_2) \tag{14}$$

$$= -(s_{2n+1}/2\pi B_1) \int_0^{2\pi} H_1^{(1)}(z_1) \sin\alpha\ se_{2n+1}(u, q)\, du \tag{15}$$

$$= (s_{2n+1}/\pi B_1)\Sigma B_{2r+1}[I_r(iw_1) K_{r+1}(-iw_2) - I_{r+1}(iw_1) K_r(-iw_2)] \tag{16}$$

$$= i^{-1}(\bar{m}_{2n+1}/\pi) w_2 \int_0^{\infty} e^{iw_1 \cosh u} \sinh u\ Se_{2n+1}(u, q)\, du \tag{17}$$

$$= -\tfrac{1}{2} Ne_{2n+1}^{(1)}(z, q) \tag{18}$$

$$Gek_{2n+2}(z, q) = -(\bar{m}_{2n+2}/\pi)\ \tanh z\ \Sigma(2r+2) B_{2r+2} K_{2r+2}(-iw_1) \qquad (b_{2n+2}) \tag{19}$$

$$= -(m_{2n+2}/\pi)\ \coth z\ \bar{\Sigma}(2r+2) B_{2r+2} K_{2r+2}(-iw_2) \tag{20}$$

$$= i(s_{2n+2}/2\pi B_2) \int_0^{2\pi} H_2^{(1)}(z_1) \sin 2\alpha\ se_{2n+2}(u, q)\, du \tag{21}$$

$$= (s_{2n+2}/\pi B_2)\Sigma B_{2r+2}[I_r(iw_1) K_{r+2}(-iw_2) - I_{r+2}(iw_1) K_r(-iw_2)] \tag{22}$$

$$= i(\bar{m}_{2n+2}/\pi) w_2 \int_0^{\infty} e^{iw_1 \cosh u} \sinh u\ Se_{2n+2}(u, q)\, du \tag{23}$$

$$= \tfrac{1}{2} i\, Ne_{2n+2}^{(1)}(z, q). \tag{24}$$

In (5), (11), (17), (23), $\cosh z > 1$ in w_1, $\sinh z > 0$ in w_2; k, z, real > 0. Since the above functions are second solutions of (3) §1, the relationships in §9 are valid if k is written for y. See §15 for the values of the functions at the origin.

11. $Me_m^{(1),(2)}(z, q)$, $Ne_m^{(1),(2)}(z, q)$, **combination solutions of** (3) §1.

$$Me_m^{(1),(2)}(z, q) = Ce_m(z, q) \pm i\, Fey_m(z, q) \tag{1, 2}$$

$$Ne_m^{(1),(2)}(z, q) = Se_m(z, q) \pm i\, Gey_m(z, q) \tag{3, 4}$$

The developments, etc., of these functions, which are useful in wave propagation problems, e.g. diffraction, may be derived from §§6, 8, by applying (1) – (4). For example

$$Me_{2n}^{(1),(2)}(z, q) = l_{2n}\Sigma A_{2r}H_{2r}^{(1),(2)}(w_1) \tag{5}$$

$$= l_{2n}\Sigma A_{2r}H_{2r}^{(1),(2)}(w_2) \tag{6}$$

$$= (p_{2n}/\pi A_0)\int_0^{2\pi} H_0^{(1),(2)}(z_1)ce_{2n}(u, q)\,du \tag{7}$$

$$= (p_{2n}/A_0)\Sigma A_{2r}J_r(v_1)H_r^{(1),(2)}(v_2). \tag{8}$$

12. $Ce_m(z, -q)$, $Se_m(z, -q)$, **first (periodic) solutions of** (4) §1.

The representations of these functions may be derived from §6 by applying the relationships:

$$Ce_{2n}(z, -q) = (-1)^n Ce_{2n}(\tfrac{1}{2}\pi i + z, q) \qquad (a_{2n}) \tag{1}$$

$$Ce_{2n+1}(z, -q) = (-1)^{n+1} i\, Se_{2n+1}(\tfrac{1}{2}\pi i + z, q) \qquad (b_{2n+1}) \tag{2}$$

$$Se_{2n+1}(z, -q) = (-1)^{n+1} i\, Ce_{2n+1}(\tfrac{1}{2}\pi i + z, q) \qquad (a_{2n+1}) \tag{3}$$

$$Se_{2n+2}(z, -q) = (-1)^{n+1} Se_{2n+2}(\tfrac{1}{2}\pi i + z, q). \qquad (b_{2n+2}) \tag{4}$$

As $q \to 0$, $Ce_m(z, -q) \to +\cosh mz$, and $Se_m(z, q) \to \sinh mz$, these being solutions of (4) §1, with $q = 0$, $a = m^2$. The multipliers $(-1)^{n+1}$ ensure the conventional positive signs. The above relations do not apply to the infinite integrals in §6.

13. $Fe_m(z, -q)$, $Ge_m(z, -q)$, **alternative second (non-periodic) solutions of** (4) §1.

$$Fe_{2n}(z, -q) = (-1)^n Re\{Fe_{2n}(\tfrac{1}{2}\pi i + z, q)\} \qquad (a_{2n}) \tag{1}$$

$$= C_{2n}(q)[z\, Ce_{2n}(z, -q) + (-1)^{n+1}\Sigma f_{2r+2}\sinh(2r+2)z] \tag{2}$$

$$Fe_{2n+1}(z, -q) = (-1)^n Im\{Ge_{2n+1}(\tfrac{1}{2}\pi i + z, q)\} \qquad (b_{2n+1}) \tag{3}$$

$$= -S_{2n+1}(q)[z\, Ce_{2n+1}(z, -q) + (-1)^{n+1}\Sigma g_{2r+1}\sinh(2r+1)z] \tag{4}$$

$$Ge_{2n+1}(z, -q) = (-1)^n Im\{Fe_{2n+1}(\tfrac{1}{2}\pi i + z, q)\} \qquad (a_{2n+1}) \tag{5}$$

$$= C_{2n+1}(q)[z\, Se_{2n+1}(z, -q) + (-1)^n\Sigma f_{2r+1}\cosh(2r+1)z] \tag{6}$$

$$Ge_{2n+2}(z, -q) = (-1)^{n+1} Re\{Ge_{2n+2}(\tfrac{1}{2}\pi i + z, q)\} \qquad (b_{2n+2}) \tag{7}$$

$$= -S_{2n+2}(q)[z\, Se_{2n+2}(z, -q) + (-1)^n\Sigma g_{2r}\cosh 2rz]. \tag{8}$$

By virtue of the relationships in §9, the behaviour of these functions as $z \to + \infty$ may be inferred from that of $Fey_m(z, -q)$, $Gey_m(z, -q)$, etc. in §28. As $q \to 0$, $Fe_m(z, -q) \to +\sin mz$, and $Ge_m(z, -q) \to +\cosh mz$, these being solutions of (4) §1, with $q = 0$, $a = m^2$. The multipliers $(-1)^{\frac{n}{n+1}}$ ensure the conventional + signs.

14. $Fey_m(z, -q)$, $Gey_m(z, -q)$, **second (non-periodic) solutions of** (4) §1. The representations of these functions may be derived from §8 by applying the relations:

$$Fey_{2n}(z, -q) = (-1)^n Fey_{2n}(\tfrac{1}{2}\pi i + z, q) \qquad (a_{2n}) \qquad (1)$$

$$Fey_{2n+1}(z, -q) = (-1)^{n+1} i\, Gey_{2n+1}(\tfrac{1}{2}\pi i + z, q) \qquad (b_{2n+1}) \qquad (2)$$

$$Gey_{2n+1}(z, -q) = (-1)^{n+1} i\, Fey_{2n+1}(\tfrac{1}{2}\pi i + z, q) \qquad (a_{2n+1}) \qquad (3)$$

$$Gey_{2n+2}(z, -q) = (-1)^{n+1} Gey_{2n+2}(\tfrac{1}{2}\pi i + z, q) \qquad (b_{2n+2}) \qquad (4)$$

The above relations do not apply to the infinite integrals in §8.

15. $Fek_m(z, -q)$, $Gek_m(z, -q)$, **second (non-periodic) solutions of** (4) §1.

$$Fek_{2n}(z, -q) = (-1)^n Fek_{2n}(\tfrac{1}{2}\pi i + z, q) \qquad (a_{2n}) \qquad (1)$$

$$= (l'_{2n}/\pi)\Sigma A_{2r} K_{2r}(w_2) \qquad (2)$$

$$= (l'_{2n}/\pi)\bar{\Sigma} A_{2r} K_{2r}(w_1) \qquad (3)$$

$$= (p'_{2n}/\pi A_0)\Sigma A_{2r} I_r(v_1) K_r(v_2) \qquad (4)$$

$$= (l_{2n}/\pi)\int_0^\infty e^{-w_1 \cosh u}\, Ce_{2n}(u, -q)\, du \qquad (5)$$

$$= \tfrac{1}{2} i\, Me_{2n}^{(1)}(z, -q). \qquad (6)$$

$$Fek_{2n+1}(z, -q) = (-1)^n Gek_{2n+1}(\tfrac{1}{2}\pi i + z, q) \qquad (b_{2n+1}) \qquad (7)$$

$$= (\bar{m}'_{2n+1}/\pi)\coth z\, \Sigma(2r+1) B_{2r+1} K_{2r+1}(w_2) \qquad (8)$$

$$= (m'_{2n+1}/\pi)\bar{\Sigma} B_{2r+1} K_{2r+1}(w_1) \qquad (9)$$

$$= (s'_{2n+1}/\pi B_1)\Sigma B_{2r+1}[I_r(v_1) K_{r+1}(v_2) - I_{r+1}(v_1) K_r(v_2)] \qquad (10)$$

$$= (m_{2n+1}/\pi)\int_0^\infty e^{-w_1 \cosh u}\, Ce_{2n+1}(u, -q)\, du \qquad (11)$$

$$= -\tfrac{1}{2} i\, Me_{2n+1}^{(1)}(z, -q). \qquad (12)$$

$$Gek_{2n+1}(z, -q) = (-1)^n Fek_{2n+1}(\tfrac{1}{2}\pi i + z, q) \qquad (a_{2n+1}) \qquad (13)$$

$$= (l'_{2n+1}/\pi)\Sigma A_{2r+1} K_{2r+1}(w_2) \qquad (14)$$

$$= (l'_{2n+1}/\pi)\tanh z\, \bar{\Sigma}(2r+1) A_{2r+1} K_{2r+1}(w_1) \qquad (15)$$

$$= (p'_{2n+1}/\pi A_1)\Sigma A_{2r+1}[I_r(v_1)K_{r+1}(v_2) + I_{r+1}(v_1)K_r(v_2)] \tag{16}$$

$$= (l_{2n+1}/\pi)w_2 \int_0^\infty e^{-w_1\cosh u} \sinh u\, Se_{2n+1}(u, -q)\, du \tag{17}$$

$$= -\tfrac{1}{2}i\, Ne^{(1)}_{2n+1}(z, -q). \tag{18}$$

$$Gek_{2n+2}(z, -q) = (-1)^{n+1}Gek_{2n+2}(\tfrac{1}{2}\pi i + z, q) \qquad (b_{2n+2}) \tag{19}$$

$$= (\bar{m}'_{2n+2}/\pi)\coth z\, \Sigma(2r+2)B_{2r+2}K_{2r+2}(w_2) \tag{20}$$

$$= (m'_{2n+2}/\pi)\tanh z\, \bar{\Sigma}(2r+2)B_{2r+2}K_{2r+2}(w_1) \tag{21}$$

$$= (s'_{2n+2}/\pi)\Sigma B_{2r+2}[I_r(v_1)K_{r+2}(v_2) - I_{r+2}(v_1)K_r(v_2)] \tag{22}$$

$$= (m_{2n+2}/\pi)w_2 \int_0^\infty e^{-w_1\cosh u} \sinh u\, Se_{2n+2}(u, -q)\, du \tag{23}$$

$$= \tfrac{1}{2}i\, Ne^{(1)}_{2n+2}(z, -q). \tag{24}$$

In (5), (11), (17), (23) $\cosh z > 1$ in w_1, $\sinh z > 0$ in w_2; k, z, real > 0. The following were deduced from Bessel function product representations:

$$Fek'_{2n}(0, -q) = -p'_{2n}l_{2n}/\pi, \tag{25}$$

$$Fek'_{2n+1}(0, -q) = -s'_{2n+1}m_{2n+1}/\pi, \tag{26}$$

$$Gek_{2n+1}(0, -q) = p'_{2n+1}l_{2n+1}/\pi, \tag{27}$$

$$Gek_{2n+2}(0, -q) = s'_{2n+2}m_{2n+2}/\pi. \tag{28}$$

16. $Me^{(1),(2)}_m(z, -q)$, $Ne^{(1),(2)}_m(z, -q)$, combination solutions of (4) §1.

$$Me^{(1),(2)}_{2n}(z, -q) = (-1)^n Me^{(1),(2)}_{2n}(\tfrac{1}{2}\pi i + z, q) = Ce_{2n}(z, -q) \pm i\, Fey_{2n}(z, -q) \qquad (a_{2n}) \tag{1}$$

$$Me^{(1),(2)}_{2n+1}(z, -q) = (-1)^{n+1} i\, Ne^{(1),(2)}_{2n+1}(\tfrac{1}{2}\pi i + z, q) = Ce_{2n+1}(z, -q) \pm i\, Fey_{2n+1}(z, -q) \qquad (b_{2n+1}) \tag{2}$$

$$Ne^{(1),(2)}_{2n+1}(z, -q) = (-1)^{n+1} i\, Me^{(1),(2)}_{2n+1}(\tfrac{1}{2}\pi i + z, q) = Se_{2n+1}(z, -q) \pm i\, Gey_{2n+1}(z, -q) \qquad (a_{2n+1}) \tag{3}$$

$$Ne^{(1),(2)}_{2n+2}(z, -q) = (-1)^{n+1} Ne^{(1),(2)}_{2n+2}(\tfrac{1}{2}\pi i + z, q) = Se_{2n+2}(z, -q) \pm i\, Gey_{2n+2}(z, -q) \qquad (b_{2n+2}) \tag{4}$$

17. $ce_{m+\beta}(z, q)$, $se_{m+\beta}(z, q)$, first and second solutions of (1) §1 of fractional order.

Fig. 1 is a stability chart* for equations (1), (2) §1[5]. a_m, b_m are character-

*The chart may be plotted from tabular values in references 2, 6, up to a_6, b_6, the q range being 0–40. By using the values given in AMP Report 165.1R, prepared by the Mathematical Tables Project, National Bureau of Standards, September 1945, the chart may be extended to a_{11}, for the q range $0 - 25$.

istic curves for $ce_m(z, \pm q)$, $se_m(z, \pm q)$, the periodic functions of integral order m. For these to be solutions, $P = (a, q)$ must lie on one of the solid curves which divide the (a, q) plane into stable and unstable regions. If P lies in an unshaded region, the solution is stable, being bounded and continuous in $z \geqq 0$,

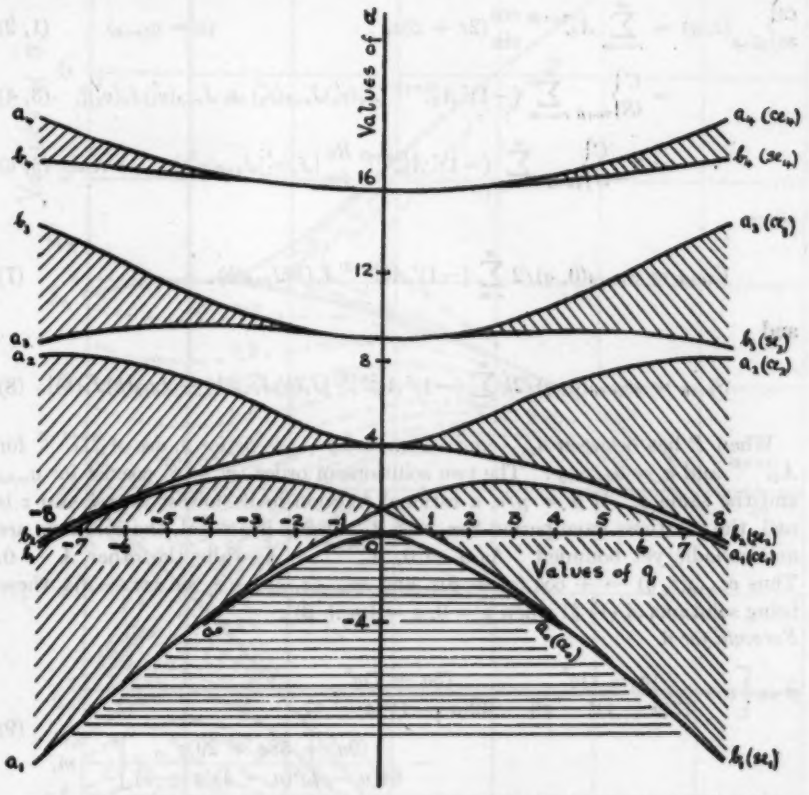

FIG. 1. Stability chart for $y'' + (a - 2q \cos 2z)y = 0$ [5]. The curves are the (a, q) characteristics for the Mathieu functions of positive integral order having period π or 2π. When (a, q) lies in an unshaded area, the solution is either periodic or non-periodic but bounded. When (a, q) lies in a shaded area the complete solution is unstable and $\to \pm \infty$ as $z \to +\infty$. A single infinity of solutions exist, however, which $\to 0$ as $z \to \infty$. These $\to \pm \infty$ as $z \to -\infty$. Note that the odd order characteristics are asymmetrical about the a axis, but the chart is symmetrical.

real. If P lies in a shaded region, the complete solution is unstable and tends to $\pm\infty$ as $z \to +\infty$. A single infinity of solutions exist, however, which tend to zero as $z \to +\infty$.

Choose P in a stable region *between*, say, a_2 and b_3. In passing from a_2 to b_3 through P in a direction parallel to the a axis, the order of the function increases

by unity. *Between* a_s and b_s the order is $(2 + \beta)$, with $0 < \beta < 1$. In general for P between a_m, b_{m+1}, the order is $(m + \beta)$, $0 < \beta < 1$ When P lies between a_{2n}, b_{2n+1}, the first (even) and second (odd) independent solutions are *defined* to be

$$\left.\begin{matrix} ce \\ se \end{matrix}\right\}_{2n+\beta}(z, q) = \sum_{r=-\infty}^{\infty} A_{2r}^{(2n+\beta)} \begin{matrix} \cos \\ \sin \end{matrix}(2r + \beta)z, \qquad (a = a_{2n+\beta}) \qquad (1, 2)$$

$$= \left.\begin{matrix} C \\ iS \end{matrix}\right\}_{2n+\beta} \sum_{r=-\infty}^{\infty} (-1)^r A_{2r}^{(2n+\beta)} [J_r(v_1')J_{r+\beta}(v_2') \pm J_{r+\beta}(v_1')J_r(v_2')], \quad (3, 4)$$

$$= 2 \left.\begin{matrix} C \\ -S \end{matrix}\right\}_{2n+\beta} \sum_{r=-\infty}^{\infty} (-1)^r A_{2r}^{(2n+\beta)} \begin{matrix} Re \\ Im \end{matrix} \{J_r(v_1')J_{r+\beta}(v_2')\}, \qquad z \text{ real} \quad (5, 6)$$

where

$$C_{2n+\beta} = ce_{2n+\beta}(0, q)/2 \sum_{-\infty}^{\infty} (-1)^r A_{2r}^{(2n+\beta)} J_r(k)J_{r+\beta}(k), \qquad (7)$$

and

$$S_{2n+\beta} = se'_{2n+\beta}(0, q)/2k \sum_{-\infty}^{\infty} (-1)^r A_{2r}^{(2n+\beta)} [J_r(k)J'_{r+\beta}(k) - J_{r+\beta}(k)J'_r(k)]. \qquad (8)$$

When P lies between a_{2n+1} and b_{2n+2}, write $(1 + \beta)$ for β, use $A_{2r+1}^{(2n+1+\beta)}$ for $A_{2r}^{(2n+\beta)}$ and $a = a_{2n+1+\beta}$. The two solutions of order $(m + \beta)$ coexist for $a_{m+\beta}$ and the same q. If $\beta = p/s$, a rational fraction in its lowest terms, and z is real, the solutions have period $2s\pi$, $s \geqq 2$. If β is irrational, the solutions are non-periodic yet bounded. As $q \to 0$, $A_m^{(m+\beta)} \to 1$, while the other $A \to 0$. Thus $ce_{m+\beta}(z, q) \to +\cos(m + \beta)z$, and $se_{m+\beta}(z, q) \to +\sin(m + \beta)z$, these being solutions of (1) §1 when $q = 0$, $a = (m + \beta)^2$.

Formula for β.

$$\beta \simeq \left[a - \frac{(a-1)q^2}{[2(a-1)^2 - q^2]} - \frac{(5a+7)q^4}{32(a-1)^3(a-4)} - \frac{(9a^2+58a+29)q^6}{64(a-1)^5(a-4)(a-9)}\right]^{\frac{1}{2}} - m, \qquad (9)$$

where m is the integral part of the order of the function. (9) gives moderate accuracy, provided the ratio of each term in [ ] to its predecessor is small enough, and no denominator vanishes. When (9) is not sufficiently accurate, or a denominator vanishes, β may be calculated by means of the recurrence relation

$$[a - (p + \beta)^2]c_p - q(c_{p+2} + c_{p-2}) = 0, \qquad (10)$$

as demonstrated in reference 9. $p = 2r$ or $2r + 1$, according as $m = 2n$ or $2n + 1$. If for a given $\nu = (m + \beta)$, the $a = a_\nu$ are computed for various $q > 0$, an iso-β curve may be plotted, as illustrated in Fig. 2. Such curves lie in the stable regions, and since (9) is even in q, they are symmetrical about the a axis.

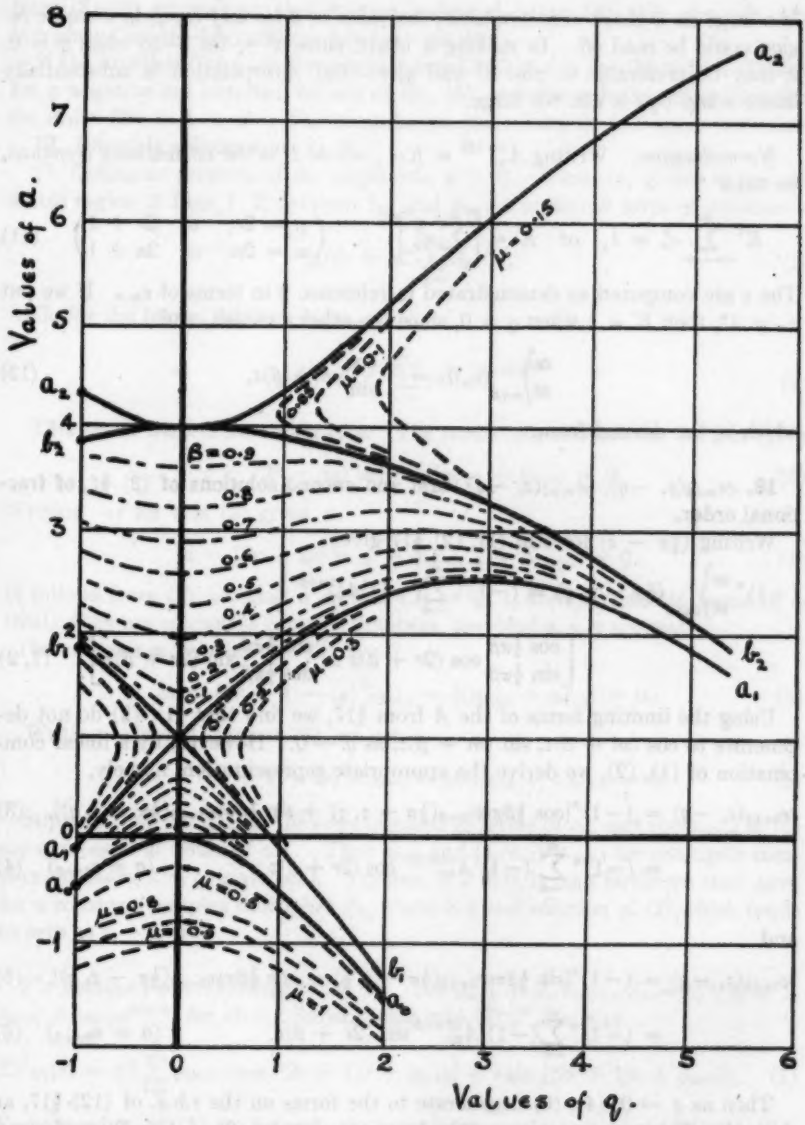

FIG. 2. Iso-$\beta\mu$ chart for $y'' + (a - 2q\cos 2z)y = 0$. The iso-$\beta$ curves pertain to stable solutions of real fractional order, $0 < \beta < 1$. If $\beta = p/s$, a proper fraction in its lowest terms, the solutions have period $2s\pi$, $s \geq 2$. If β is irrational the solutions are non-periodic but bounded. The iso-μ curves pertain to the unstable solutions (see underline to Fig. 1).

If a large iso-β chart were available, the value of β for any (a, q) in a stable region could be read off. In making a chart, since $a^{\frac{1}{2}} = (m + \beta)$ when $q = 0$, it may be preferable to plot $a^{\frac{1}{2}}$ and q, so that interpolation is substantially linear when $|q|$ is not too large.

Normalisation. Writing $A_p^{(m+\beta)} = Kc_p$, where K is the normalising constant, we take

$$K^2 \sum_{r=-\infty}^{\infty} c_p^2 = 1, \quad \text{or} \quad K = \left[\sum_{-\infty}^{\infty} c_p^2\right]^{-\frac{1}{2}} \qquad \begin{pmatrix} p = 2r, & \text{or} & 2r+1 \\ m = 2n, & \text{or} & 2n+1 \end{pmatrix} \tag{11}$$

The c are computed as demonstrated in reference 9 in terms of c_m. If we put $c_m = 1$*, then $K = 1$ when $q = 0$, since the other c vanish, and

$$\left.\begin{matrix} ce \\ se \end{matrix}\right\}_{m+\beta}(z, 0) = +\begin{matrix} \cos \\ \sin \end{matrix}(m + \beta)z, \tag{12}$$

which is the desired form.

18. $ce_{m+\beta}(z, -q)$, $se_{m+\beta}(z, -q)$, first and second solutions of (2) §1, of fractional order.

Writing $(\frac{1}{2}\pi - z)$ for z in (1), (2) §17 gives

$$(-1)^n \left.\begin{matrix} ce \\ se \end{matrix}\right\}_{2n+\beta}(\tfrac{1}{2}\pi - z, q) = (-1)^n \sum_{-\infty}^{\infty}(-1)^r A_{2r}^{(2n+\beta)}$$

$$\left[\begin{matrix} \cos \frac{1}{2}\pi\beta \\ \sin \frac{1}{2}\pi\beta \end{matrix} \cos (2r + \beta)z \pm \begin{matrix} \sin \frac{1}{2}\pi\beta \\ \cos \frac{1}{2}\pi\beta \end{matrix} \sin (2r + \beta)z\right]. \tag{1, 2}$$

Using the limiting forms of the A from §17, we find that (1), (2) do not degenerate to $\cos (m + \beta)z$, $\sin (m + \beta)z$, as $q \to 0$. However, by a linear combination of (1), (2), we derive the appropriate representation, namely,

$$ce_{2n+\beta}(z, -q) = (-1)^n[\cos \tfrac{1}{2}\beta\pi ce_{2n+\beta}(\tfrac{1}{2}\pi - z, q) + \sin \tfrac{1}{2}\beta\pi se_{2n+\beta}(\tfrac{1}{2}\pi - z, q)], \tag{3}$$

$$= (-1)^n \sum_{-\infty}^{\infty} (-1)^r A_{2r}^{(2n+\beta)} \cos (2r + \beta)z\,; \qquad (a = a_{2n+\beta}) \tag{4}$$

and

$$se_{2n+\beta}(z, -q) = (-1)^n[\sin \tfrac{1}{2}\beta\pi ce_{2n+\beta}(\tfrac{1}{2}\pi - z, q) - \cos \tfrac{1}{2}\beta\pi se_{2n+\beta}(\tfrac{1}{2}\pi - z, q)], \tag{5}$$

$$= (-1)^n \sum_{-\infty}^{\infty}(-1)^r A_{2r}^{(2n+\beta)} \sin (2r + \beta)z. \qquad (a = a_{2n+\beta}) \tag{6}$$

Then as $q \to 0$, (4), (6) degenerate to the forms on the r.h.s. of (12) §17, as desired. The representations of the functions of order $(2n + 1 + \beta)$ are derived

* The c have zeros. Thus if c_m had a zero for a particular value of q in a computation, another coefficient would be taken as unity.

from (3)–(6) by making the changes indicated below (8) §17: also $\sin \frac{1}{2}\beta\pi$ is replaced by $\cos \frac{1}{2}\beta\pi$, and $\cos \frac{1}{2}\beta\pi$ by $-\sin \frac{1}{2}\beta\pi$.

If the functions for q positive are tabulated in $0 \leqq z \leqq \frac{1}{2}\pi$, the values of those for q negative are obtained by aid of (3), (5), and the corresponding formulae for order $(2n + 1 + \beta)$. The remarks on periodicity in §17 apply here also.

19. Unstable solutions of *(1)* §*1*.

1°. Conjugate property of the coefficients, $q > 0$. When (a, q) lies in an unstable region of Figs. 1, 2, between b_{2n} and a_{2n}, a preferred form of solution is

$$y_1(z) = e^{\mu z} \sum_{-\infty}^{\infty} c_{2r} e^{2rzi}, \tag{1}$$

while for the region between b_{2n+1}, a_{2n+1},

$$y_1(z) = e^{\mu z} \sum_{-\infty}^{\infty} c_{2r+1} e^{(2r+1)zi}. \tag{2}$$

This choice ensures the reality of μ. The recurrence relation for the c in (1) is

$$[a - (2r - i\mu)^2]c_{2r} - q(c_{2r+2} + c_{2r-2}) = 0. \tag{3}$$

Writing $-r$ for r in (3) gives

$$[a - (2r + i\mu)^2]c_{-2r} - q(c_{-2r-2} + c_{-2r+2}) = 0. \tag{4}$$

It follows from (3), (4) that if c_{2r} and c_{-2r}, $r \geqq 1$, are expressed in terms of c_0 (real), they are conjugate complex numbers, provided a, q, μ are real.

The recurrence relation for (2) is

$$[a - (2r + 1 - i\mu)^2]c_{2r+1} - q(c_{2r+3} + c_{2r-1}) = 0. \tag{5}$$

Writing $-(r + 1)$ for r in (5), we get

$$[a - (2r + 1 + i\mu)^2]c_{-2r-1} - q(c_{-2r-3} + c_{-2r+1}) = 0. \tag{6}$$

Suppose that the c_{2r+1} in (5) are expressed in terms of c_1, and the c_{-2r-1} in (6) are expressed in terms of c_{-1}. Then c_{2r+1} and $(c_1/c_{-1})^* c_{-2r-1}$ are conjugate complex numbers if a, q, μ are real. Further, if $\mu > 0$, it may be shown that, save for a constant complex multiplier Z_0, there is a real solution of (2) which tends to zero as $z \to -\infty$.

2°. Solutions corresponding to (2) 1°. Let $c_{2r+1} = Z_0 d_{2r+1}$, $Z_0 = e^{i\vartheta_0}$, $c_1 = 1$, $d_{2r+1} = \rho_{2r+1} e^{i\varphi_{2r+1}}$ for all r. Substituting into (2) *1°*, leads to

$$Z_0^{-1} y_1(z) = e^{\mu z} \sum_{-\infty}^{\infty} \rho_{2r+1} \{\cos[(2r + 1)z + \varphi_{2r+1}] + i \sin[(2r + 1)z + \varphi_{2r+1}]\}. \tag{1}$$

For the r.h.s. of (1) to be real, the imaginary part must vanish identically, and this means that

* c_1/c_{-1} is obtained in 3° below.

$$\rho_{2r+1} = \rho_{-2r-1}, \quad \text{and} \quad \varphi_{2r+1} = -(\varphi_{-2r-1} + 2s\pi), \tag{2}$$

where s is integral. Hence d_{2r+1} and d_{-2r-1} are conjugate, so (1) may be written in the form

$$\tfrac{1}{2}Z_0^{-1}y_1(z) = e^{sz}\Sigma\rho_{2r+1}\cos[(2r+1)z + \varphi_{2r+1}]. \tag{3}$$

A second independent solution is obtained by writing $-z$ for z, so we get

$$\tfrac{1}{2}Z_0^{-1}y_2(z) = e^{-sz}\Sigma\rho_{2r+1}\cos[2r+1)z - \varphi_{2r+1}]. \tag{4}$$

Then (3) $\to 0$ as $z \to -\infty$, while (4) $\to \pm\infty$.

3°. Determination of Z_0. Since d_{2r+1} and d_{-2r-1} are conjugate, it follows that, $|Z_0|$ being unity,

$$|c_{2r+1}| = |c_{-2r-1}|. \tag{1}$$

Thus we may write $c_{2r+1} = \rho_{2r+1}e^{i\psi_{2r+1}}$, $c_{-2r-1} = \rho_{2r+1}e^{i\psi_{-2r-1}}$. Then

$$Z_0 = e^{i\theta_0} = c_{2r+1}/d_{2r+1} = e^{i(\psi_{2r+1}-\varphi_{2r+1})} \tag{2}$$

$$= c_{-2r-1}/d_{-2r-1} = e^{i(\psi_{-2r-1}+\varphi_{2r+1}+2s\pi)} \tag{3}$$

by (2), 2°. Hence by (2), (3)

$$\theta_0 = \tfrac{1}{2}(\psi_{2r+1} + \psi_{-2r-1}) + s\pi. \tag{4}$$

Since $c_1 = 1$, $\psi_1 = 0$, so that in 1° above, $c_1/c_{-1} = e^{-2i\theta_0}$. (5)

4°. Solutions corresponding to (1) 1°. It may be shown that these are

$$\tfrac{1}{2}y_1(z) = e^{sz}\left\{\rho_0 + \sum_{r=1}^{\infty}\rho_{2r}\cos(2rz + \varphi_{2r})\right\}, \tag{1}$$

and

$$\tfrac{1}{2}y_2(z) = e^{-sz}\left\{\rho_0 + \sum_{r=1}^{\infty}\rho_{2r}\cos(2rz - \varphi_{2r})\right\}. \tag{2}$$

Here $c_{2r} = \rho_{2r}e^{i\varphi_{2r}}$, $c_0 = 2\rho_0$ real, $\rho_{2r} = \rho_{-2r}$, $\varphi_{2r} = -(\varphi_{-2r} + 2s\pi)$, so for $r \geqq 1$ c_{2r} and c_{2r} are conjugate; also $Z_0 = 1$.

20. $ceu_{m+p}(\pm z, q)$, first and second solutions of (1) §1, for an unstable region of the (a, q) plane, q positive.

When (a, q) lies between b_{2n} and a_{2n} in Figs. 1, 2, we take (see 4° §19)

$$ceu_{2n+p}(\pm z, q) = Ke^{\pm sz}\left\{\rho_0 + \sum_{r=1}^{\infty}\rho_{2r}\cos(2rz \pm \varphi_{2r})\right\}, \quad (b_{2n+p}) \quad (1, 2)$$

and between b_{2n+1}, a_{2n+1}, we take (see 2° §19)

$$ceu_{2n+1+p}(\pm z, q) = K_1e^{\pm sz}\Sigma\rho_{2r+1}\cos[(2r+1)z \pm \varphi_{2r+1}], \quad (b_{2n+1+p}) \quad (3, 4)$$

where K, K_1 are normalising constants defined in §24. The u in ceu signifies that (a, q) lies in an unstable region of Figs. 1, 2.

Alternative definitions. Taking half the sum of (1), (2) yields the *even* solution.

$$\mathfrak{y}_1(z) = K\Big\{\cosh \mu z\Big[\rho_0 + \sum_1^\infty \rho_{2r} \cos 2rz \cos \varphi_{2r}\Big] - \sinh \mu z \sum_1^\infty \rho_{2r} \sin 2rz \sin \varphi_{2r}\Big\}. \quad (b_{2n+\mu}) \qquad (5)$$

Half the difference between (1), (2) yields the *odd* solution

$$\mathfrak{y}_2(z) = K\{\sinh \mu z[\text{as at (5)}] - \cosh \mu z \Sigma(\text{as at 5})\}. \quad (b_{2n+\mu}) \qquad (6)$$

Similarly from (3), (4) we obtain the *even* solution

$$\mathfrak{y}_1(z) = K_1\{\cosh \mu z \Sigma \rho_{2r+1} \cos (2r+1)z \cos \varphi_{2r+1} - \sinh \mu z \Sigma \rho_{2r+1} \sin (2r+1)z \sin \varphi_{2r+1}\}, \quad (b_{2n+1+\mu}) \qquad (7)$$

and the *odd* solution

$$\mathfrak{y}_2(z) = K_1\{\sinh \mu z \Sigma(\text{as at 7}) - \cosh \mu z \Sigma(\text{as at 7})\}. \quad (b_{2n+1+\mu}) \qquad (8)$$

Comment on (1)–(4), and (5)–(8). In applications the former are preferable to the latter for the following reasons:

(a) One solution tends to zero, the other to infinity with z, whereas the even and odd solutions *both* tend to infinity as $z \to +\infty$.

(b) In numerical work it is expedient to use tables of $e^{\pm z}$, rather than those of $\sinh z$, $\cosh z$.

(c) Analytical work is likely to be simpler.

Functions of complex order $m - i\mu$. If in the representations of $\left.\begin{matrix} ce \\ se \end{matrix}\right\}_{2n+1+\beta}(z, q)$ we write $-i\mu$ for β*, we obtain

$$\left.\begin{matrix} ce \\ i\,se \end{matrix}\right\}_{2n+1-i\mu}(z, q) = \sum_{-\infty}^{\infty} A_{2r+1}^{(2n+1-i\mu)} \begin{matrix} \cos \\ \sin \end{matrix} (2r+1-i\mu)z. \qquad (9, 10)$$

Then by substituting

$$Z_0^{-1}[A_{2r+1}^{(2n+1-i\mu)} + A_{-2r-1}^{(2n+1-i\mu)}] = K_1 \rho_{2r+1} \cos \varphi_{2r+1}, \qquad (11)$$

and

$$i^{-1} Z_0^{-1}[A_{2r+1}^{(2n+1-i\mu)} - A_{-2r-1}^{(2n+1-i\mu)}] = K_1 \rho_{2r+1} \sin \varphi_{2r+1}, \qquad (12)$$

in (9), (10), we obtain the even and odd solutions (7), (8). Since μ is real, $(2n+1-i\mu)$ is complex and, therefore, (9), (10) may be designated Mathieu functions of complex order.

21. Computation of μ and the c. The method is described in reference 9. Knowing the c, the values of ρ and φ may then be calculated. In calculations pertaining to a function of order $m + \mu$, it may be preferable to put $c_m = 1$, rather than c_0 or $c_1 = 1$. If μ is constant and the corresponding a for different q

* It should be noted that the range of β is $0 < \beta < 1$, whereas $\mu > 0$.

are computed, an iso-μ curve of the type illustrated in Fig. 2 may be plotted in the appropriate unstable region. On b_m, a_m, $\mu = 0$, while at some point on the segment of a line joining them and parallel to the a axis, μ attains a maximum value. This is true also for points on the segment of a line parallel to the q axis and terminating on a_m $(q > 0)$. Then if the turning point (nearest the a axis) on an iso-μ curve is $q = q_0 > 0$, there are two values of a for any $q > q_0$ on the curve. Moreover, the coefficients in the solutions corresponding to the two a are different, except when $q = q_0$. Given the numerical data, families of iso-μ curves could be plotted in the unstable regions. Tabulation of a, $a^{\frac{1}{2}}$, q, μ would permit the value of the latter being found directly or by interpolation. The solution of a Mathieu equation with (a, q) in a 'charted' unstable region would then be completed by using the procedure given in reference 9.

Discrimination between solutions for the same q but different a; and vice-versa. When q is positive, on the upper and lower parts, and at the turning point of an iso-μ curve, the symbols

$$ce_{m+\bar{\mu}}, \quad ce_{m+\underline{\mu}}, \quad ce_{m+\hat{\mu}}, \text{ respectively,} \tag{1}$$

may be employed. Symbolism for solutions having the same a but different q is left for the reader to devise.

22. Degenerate forms, q positive. If a remains constant as (a, q) moves towards a_{2n} in Fig. 2 (in either direction), $\bar{\mu} \to 0$ and in (1), (2) §20, $K\bar{\rho}_0 \to A_0^{(2n)}$, $K\bar{\rho}_{2r} \to A_{2r}^{(2n)}$ and $\bar{\varphi}_{2r} \to 0$, $r \geqq 1$. When $\bar{\mu} = 0$, by proper choice of K we obtain $ce_{2n}(z, q)$. As (a, q) moves towards b_{2n}, $\underline{\mu} \to 0$, and $\underline{\rho}_0 \to 0$, $K\underline{\rho}_{2r} \to B_{2r}^{(2n)}$, $\underline{\varphi}_{2r} \to -\frac{1}{2}\pi$, $r \geqq 1$, so with $\underline{\mu} = 0$ and proper choice of K, we get $\pm se_{2n}(z, q)$. Similarly with appropriate K_1 the degenerate forms of (3), (4) §20 are $ce_{2n+1}(z, q)$ and $\pm se_{2n+1}(z, q)$. These functions may be shown to be the degenerate forms of (5)–(8) §20, each to each.

23. Functions of order $(m + \mu)$, q negative. These are defined as follows:

$$ceu_{2n+\mu}(\pm z, q) = (-1)^n e^{\mp \frac{1}{2}\pi\mu} ce_{2n+\mu}[\pm(\tfrac{1}{2}\pi - z), q], \qquad (b_{2n+\mu}) \qquad (1, 2)$$

$$= (-1)^n K e^{\mp\mu z}\left\{\rho_0 + \sum_{r=1}^{\infty} (-1)^r \rho_{2r} \cos(2rz \mp \varphi_{2r})\right\}: \qquad (3, 4)$$

$$ceu_{2n+1+\mu}(\pm z, q) = (-1)^n e^{\mp \frac{1}{2}\pi\mu} ce_{2n+1+\mu}[\pm(\tfrac{1}{2}\pi - z), q], \qquad (a_{2n+1+\mu}) \qquad (5, 6)$$

$$= (-1)^n K_1 e^{\mp\mu z} \Sigma \rho_{2r+1} [\sin(2r + 1)z \mp \varphi_{2r+1}]. \qquad (7, 8)$$

The r.h.s. of (7), (8) may be expressed in a cosine series by altering the argument to $[(2r + 1)z - \frac{1}{2}\pi \mp \varphi_{2r+1}]$. The definition would then be in keeping with the notation ce_μ.

Degenerate forms. When $\bar{\mu} = 0$ in (3), (4), (7), (8), the functions degenerate to $ce_{2n}(z, -q)$, $se_{2n+1}(z, -q)$. When $\underline{\mu} = 0$, the functions degenerate to $\mp se_{2n}(z, -q)$, $\pm ce_{2n+1}(z, -q)$. These forms are to be expected from Fig. 2,

$q < 0$. The multiplier $(-1)^n$ ensures that the sign of the degenerate form is in accordance with the definitions (1), (5), (10), (15) §4, provided the upper sign is chosen when $\mu = 0$.

24. Normalisation. The rules must be such that the degenerate forms of the functions of order $m + \mu$, when $\mu = 0$, are those given in §§22, 23. Then

$$K^2\left[2\rho_0^2 + \sum_1^\infty \rho_{2r}^2\right] = 1, \quad \text{or} \quad K = 1\Big/\left[2\rho_0^2 + \sum_1^\infty \rho_{2r}^2\right]^{\frac{1}{2}}, \tag{1}$$

and

$$K_1^2 \Sigma \rho_{2r+1}^2 = 1, \quad \text{or} \quad K_1 = 1/[\Sigma \rho_{2r+1}^2]^{\frac{1}{2}}. \tag{2}$$

25. $Ce_{m+\beta}(z, \pm q)$, $Se_{m+\beta}(z, \pm q)$, **first and second solutions of** (3), (4) §1, **of fractional order.** Series may be derived from (1), (2) §17; (4), (6) §18 in conjunction with the definitions of $\left.\begin{matrix} ce \\ se \end{matrix}\right\}_{2n+1+\beta}(z, q)$ obtainable from these sections, by writing iz for z, and omitting the external multiplier i. If $\beta = p/s$, a rational fraction in its lowest terms, the functions have period $2s\pi i$, $s \geqq 2$. As $q \to 0$, $a \to (m + \beta)^2$, and $Ce_{m+\beta}(z, \pm q) \to +\cosh(m + \beta)z$, $Se_{m+\beta}(z, \pm q) \to +\sinh(m + \beta)z$.

26. $cer_m z$, $cei_m z$, $ser_m z$, $sei_m z$. In the problem of eddy currents in a cylinder of elliptical cross-section, q is negative imaginary. Then in (1) §1, if $q = -is$, where s is real and positive*, $ce_m(z, -is)$ is a complex function of z. Thus we may separate it into real and imaginary parts by writing

$$ce_m(z, -is) = cer_m z + i\, cei_m z. \tag{1}$$

If we define

$$ce_{2n}(z, is) = \Sigma A_{2r} \cos 2rz, \tag{2}$$

the A are complex and may be computed as shown in reference 4. Then if we write $(\frac{1}{2}\pi - z)$ for z in (2), by (1) §4 we obtain

$$ce_{2n}(z, -is) = (-1)^n \Sigma A_{2r} \cos 2rz. \tag{3}$$

Taking $A_p = |A_p| e^{i\phi_p}$ in (3), we get

$$\left.\begin{matrix} cer \\ cei \end{matrix}\right\}_{2n}(z) = (-1)^n \Sigma |A_{2r}| \begin{matrix} \cos \\ \sin \end{matrix} (\phi_{2r}) \cos 2rz. \tag{4, 5}$$

Similarly with $B_p = |B_p| e^{i\chi_p}$, we have

$$\left.\begin{matrix} cer \\ cei \end{matrix}\right\}_{2n+1}(z) = (-1)^n \Sigma |B_{2r+1}| \begin{matrix} \cos \\ \sin \end{matrix} (\chi_{2r+1}) \cos (2r + 1)z, \tag{6, 7}$$

$$\left.\begin{matrix} ser \\ sei \end{matrix}\right\}_{2n+1}(z) = (-1)^n \Sigma |A_{2r+1}| \begin{matrix} \cos \\ \sin \end{matrix} (\phi_{2r+1}) \sin (2r + 1)z \tag{8, 9}$$

* In this section s is not necessarily integral.

$$\left.\begin{matrix} ser \\ sei \end{matrix}\right\}_{2n+2}(z) = (-1)^n \Sigma \mid B_{2r+2} \mid \begin{matrix} \cos \\ \sin \end{matrix} (\chi_{2r+2}) \sin (2r+2)z. \qquad (10, 11)$$

These functions may be expressed in terms of the polar form of the *ber* and *bei* functions as follows: Writing $(is)^{\frac{1}{2}} = i^{\frac{1}{2}}l$ for k in (3) §4, using the complex A from above, and $I_{2r}(u) = (-1)^r J_{2r}(iu)$, we get

$$ce_{2n}(z, -is) = (-1)^n \frac{ce_{2n}(0, is)}{A_0} \Sigma A_{2r} J_{2r}(2i^{3/2} l \cos z). \qquad (12)$$

By reference 8, pp. 121, 122,

$$J_{2r}(i^{3/2}u) = ber_{2r}u + i\, bei_{2r}u = M_{2r}(u)e^{i\theta_{2r}(u)}. \qquad (13)$$

Putting

$$ce_{2n}(0, is)/A_0 = E_{2n}e^{i\psi_{2n}} \qquad (14)$$

and substituting from (13), (14) into (12) leads to

$$ce_{2n}(z, -is) = (-1)^n E_{2n}\Sigma \mid A_{2r} \mid M_{2r}(2l \cos z)e^{i(\theta_{2r}+\phi_{2r}+\psi_{2r})}. \qquad (15)$$

Hence

$$\left.\begin{matrix} cer \\ cei \end{matrix}\right\}_{2n}(z) = (-1)^n E_{2n}\Sigma \mid A_{2r} \mid M_{2r}(2l \cos z) \begin{matrix} \cos \\ \sin \end{matrix} (\theta_{2r} + \phi_{2r} + \psi_{2r}). \qquad (16, 17)$$

Series for $cer_{2n+1}z$, $cei_{2n+1}z$, $ser_m z$, $sei_m z$ may be derived in a similar way. Representations may also be found in terms of M functions with argument $2l \sin z$, and in Bessel function product series. The representations of the modified *cer*, *cei*, etc., functions, namely, $Cer_m z$, $Cei_m z$, $Ser_m z$, $Sei_m z$ are derived from those given above by writing iz for z.

27. Asymptotic expansions for z large. The following abbreviations are used:

$$R_m^{(a),(b)} = (2/\pi v_2)^{\frac{1}{2}}[P_m^{(a),(b)} \sin (v_2 + \tfrac{1}{4}\pi) - Q_m^{(a),(b)} \cos (v_2 + \tfrac{1}{4}\pi)] \qquad (1)$$

$$S_m^{(a),(b)} = (2/\pi v_2)^{\frac{1}{2}}[P_m^{(a),(b)} \cos (v_2 + \tfrac{1}{4}\pi) + Q_m^{(a),(b)} \sin (v_2 + \tfrac{1}{4}\pi)] \qquad (2)$$

$$P = 1 - c_2 v_2^{-2} + c_4 v_2^{-4} - \cdots \qquad (3)$$

$$Q = c_1 v_2^{-1} - c_3 v_2^{-3} + c_5 v_2^{-5} - \cdots \qquad (4)$$

$$c_1 = -(4a - 1^2)/8;\ c_2 = (4a - 1^2)(4a - 3^2)/2!8^2 \qquad (5)$$

$$c_3 = -[(4a - 1^2)(4a - 3^2)(4a - 5^2)/3!8^3] - k^4/3! \qquad (6)$$

$$c_4 = [(4a - 1^2)(4a - 3^2) \cdots (4a - 7^2)/4!8^4] + k^4(4a - 13)/2.4! \qquad (7)$$

$$2(r+1)c_{r+1} + [a - (r + \tfrac{1}{2})^2]c_r + k^4 c_{r-2} = 0. \qquad (r \geqq 2) \qquad (8)$$

$$\left.\begin{matrix} T \\ U \end{matrix}\right\}_m^{(a),(b)} = (2/\pi v_2)^{\frac{1}{2}}[e^{v_2}\Sigma c_r v_2^{-r} \mp ie^{-v_2}\Sigma c_r v_2^{-r}]. \qquad (9, 10)$$

$$X_m^{(a),(b)} = (1/2\pi v_2)^{\frac{1}{2}} e^{-v_2}\Sigma c_r v_2^{-r}; \qquad Y = e^{i(v_2+\frac{1}{4}\pi)}/(2\pi v_2)^{\frac{1}{2}}. \qquad (11)$$

$$Z = e^{v_2}/(2\pi v_2)^{\frac{1}{2}}; \qquad \bar{Z} = e^{-v_2}/(2\pi v_2)^{\frac{1}{2}}. \tag{12}$$

$P_m^{(a)}$ means that in the series for P, the c are computed using $a = a_m$, e.g. for $Ce_{2n}(z, q)$, a_{2n} would be used. In numerical work terms in k should be small compared with those devoid of k.† The latter are the same as the coefficients in the asymptotic expansions of the J- and Y-Bessel functions.

Expansions.

$$Ce_{2n}(z, q) \sim p_{2n} R^{(a)}_{2n} \tag{13}$$

$$Ce_{2n+1}(z, q) \sim -\, p_{2n+1} S^{(a)}_{2n+1} \tag{14}$$

$$Se_{2n+1}(z, q) \sim -\, s_{2n+1} S^{(b)}_{2n+1} \tag{15}$$

$$Se_{2n+2}(z, q) \sim s_{2n+2} R^{(b)}_{2n+2} \tag{16}$$

$$Fey_{2n}(z, q) \sim -\, p_{2n} S^{(a)}_{2n} \tag{17}$$

$$Fey_{2n+1}(z, q) \sim -\, p_{2n+1} R^{(a)}_{2n+1} \tag{18}$$

$$Gey_{2n+1}(z, q) \sim -\, s_{2n+1} R^{(b)}_{2n+1} \tag{19}$$

$$Gey_{2n+2}(z, q) \sim -\, s_{2n+2} S^{(b)}_{2n+2}. \tag{20}$$

Formulae for $Fek_m(z, q)$, $Gek_m(z, q)$ may be derived by applying (6), (12), (18) (24) § 10 to (13)–(20) above.

Applying the relationships in §12 to (13) – (16) yields

$$Ce_{2n}(z, -q) \sim p'_{2n} T^{(a)}_{2n} \tag{21}$$

$$Ce_{2n+1}(z, -q) \sim s'_{2n+1} U^{(b)}_{2n+1}. \tag{22}$$

$$Se_{2n+1}(z, -q) \sim p'_{2n+1} U^{(a)}_{2n+1} \tag{23}$$

$$Se_{2n+2}(z, -q) \sim s'_{2n+2} T^{(b)}_{2n+2} \tag{24}$$

These functions are real if z is real, so the imaginary parts in T, U must then be omitted. They are included when z is complex, $-\frac{1}{2}\pi < \text{phase } z < \frac{1}{2}\pi$.

Applying the relationships in 14 § to (17)–(20) leads to

$$Fey_{2n}(z, -q) \sim i\, p'_{2n} U^{(a)}_{2n} \tag{25}$$

$$Fey_{2n+1}(z, -q) \sim i\, s'_{2n+1} T^{(b)}_{2n+1} \tag{26}$$

$$Gey_{2n+1}(z, -q) \sim i\, p'_{2n+1} T^{(a)}_{2n+1} \tag{27}$$

$$Gey_{2n+2}(z, -q) \sim i\, s'_{2n+2} U^{(b)}_{2n+2} \tag{28}$$

Applying (6), (12), (18), (24) § 15 to (21)–(28) yields

$$Fek_{2n}(z, -q) \sim p'_{2n} X^{(a)}_{2n} \tag{29}$$

$$Fek_{2n+1}(z, -q) \sim s'_{2n+1} X^{(b)}_{2n+1} \tag{30}$$

$$Gek_{2n+1}(z, -q) \sim p'_{2n+1} X^{(a)}_{2n+1} \tag{31}$$

$$Gek_{2n+1}(z, -q) \sim s'_{2n+2} X^{(b)}_{2n+2} \tag{32}$$

† If v_2 is large enough, this restriction is unnecessary.

The asymptotic expansions for $Me_m^{(1),(2)}(z, \pm q)$ and $Ne_m^{(1),(2)}(z, \pm q)$ may be derived from the foregoing by applying the relationships in §§ 11, 16. The phase variation of z in all these formulae is $-\frac{1}{2}\pi < \text{phase } z < \frac{1}{2}\pi$.

28. Dominant terms in asymptotic expansions. These are useful in determining the behaviour of the functions as $z \to +\infty$.

$$Ce_{2n}(z, q) \sim p_{2n}(2/\pi v_2)^{\frac{1}{2}} \sin(v_2 + \tfrac{1}{4}\pi) \tag{1}$$

$$Ce_{2n+1}(z, q) \sim -p_{2n+1}(2/\pi v_2)^{\frac{1}{2}} \cos(v_2 + \tfrac{1}{4}\pi) \tag{2}$$

$$Se_{2n+1}(z, q) \sim -s_{2n+1}(2/\pi v_2)^{\frac{1}{2}} \cos(v_2 + \tfrac{1}{4}\pi) \tag{3}$$

$$Se_{2n+2}(z, q) \sim s_{2n+2}(2/\pi v_2)^{\frac{1}{2}} \sin(v_2 + \tfrac{1}{4}\pi) \tag{4}$$

$$Fey_{2n}(z, q) \sim -p_{2n}(2/\pi v_2)^{\frac{1}{2}} \cos(v_2 + \tfrac{1}{4}\pi) \tag{5}$$

$$Fey_{2n+1}(z, q) \sim -p_{2n+1}(2/\pi v_2)^{\frac{1}{2}} \sin(v_2 + \tfrac{1}{4}\pi) \tag{6}$$

$$Gey_{2n+1}(z, q) \sim -s_{2n+1}(2/\pi v_2)^{\frac{1}{2}} \sin(v_2 + \tfrac{1}{4}\pi) \tag{7}$$

$$Gey_{2n+2}(z, q) \sim -s_{2n+2}(2/\pi v_2)^{\frac{1}{2}} \cos(v_2 + \tfrac{1}{4}\pi) \tag{8}$$

As $z \to +\infty$, all these functions alternate with increasing rapidity, while their amplitudes tend exponentially to zero.

$$Fek_{2n}(z, q) \sim p_{2n}Y \tag{9}$$

$$Fek_{2n+1}(z, q) \sim p_{2n+1}Y \tag{10}$$

$$Gek_{2n+1}(z, q) \sim s_{2n+1}Y \tag{11}$$

$$Gek_{2n+2}(z, q) \sim s_{2n+2}Y \tag{12}$$

(9)–(12) As $z \to +\infty$, the real and imaginary parts of these functions alternate with increasing rapidity, while their amplitudes tend exponentially to zero.

$$Ce_{2n}(z, -q) \sim p'_{2n}Z \tag{13}$$

$$Ce_{2n+1}(z, -q) \sim s'_{2n+1}Z \tag{14}$$

$$Se_{2n+1}(z, -q) \sim p'_{2n+1}Z \tag{15}$$

$$Se_{2n+2}(z, -q) \sim s'_{2n+2}Z \tag{16}$$

(13)–(16) As $z \to +\infty$, all these functions tend exponentially to infinity.

$$Fey_{2n}(z, -q) \sim i\,p'_{2n}Z \tag{17}$$

$$Fey_{2n+1}(z, -q) \sim i\,s'_{2n+1}Z \tag{18}$$

$$Gey_{2n+1}(z, -q) \sim i\,p'_{2n+1}Z \tag{19}$$

$$Gey_{2n+2}(z, -q) \sim i\,s'_{2n+2}Z \tag{20}$$

(17)–(20) Each function is i times the corresponding function in (13)–(16), so as $z \to +\infty$ they tend exponentially to $+i\infty$.

$$Fek_{2n}(z, -q) \sim p'_{2n}\overline{Z} \tag{21}$$

$$Fek_{2n+1}(z, -q) \sim s'_{2n+1}\overline{Z} \tag{22}$$

(21)–(22) As $z \to +\infty$, all these functions tend exponentially to zero.

$$Gek_{2n+1}(z, -q) \sim p'_{2n+1}\overline{Z} \qquad (23)$$

$$Gek_{2n+2}(z, -q) \sim s'_{2n+2}\overline{Z} \qquad (24)$$

The dominant terms in the asymptotic formulae for $Me_m^{(1),(2)}(z, \pm q)$, and $Ne_m^{(1),(2)}(z, \pm q)$ may be derived from the foregoing by applying the relationships in §§ 11, 16.

29. Real zeros of $Ce_m(z, q)$, $Se_m(z, q)$, $Fey_m(z, q)$, $Gey_m(z, q)$, z large and positive. Since $v_2 = ke^z$, if z is large real and positive, and k is such that v_2 is large enough, the zeros in both z and $q = k^2$ may be derived from the formulae in § 28. For example by (1) § 28 the zeros of $Ce_{2n}(z, q)$ occur when

$$\sin(v_2 + \tfrac{1}{4}\pi) \simeq 0, \text{ or } v_2 \simeq \pi(s - \tfrac{1}{4}), \qquad (1)$$

where s is an integer such that v_2 is large enough for validity of the asymptotic approximation. Thus from (1) we get

$$z \simeq \log[\pi(s - \tfrac{1}{4})/k], \qquad (2)$$

and

$$q \simeq \pi^2(s - \tfrac{1}{4})^2 e^{-2z}. \qquad (3)$$

Hence if $v_2 = ke^z$ is constant (but large enough) as $z \to +\infty$, it follows from (3) that the zeros in q condense in the neighborhood of $q = 0$.

30. Asymptotic formulae for q large. If $q = k^2$ is large and *positive*, and $-\frac{1}{2}\pi < z < \frac{1}{2}\pi$, then

$$\begin{matrix} ce_m \\ se_{m+1} \end{matrix}(z, q) \sim \begin{matrix} C_m \\ S_{m+1} \end{matrix} \Big\{ e^{v_2}[\cos(\tfrac{1}{2}z + \tfrac{1}{4}\pi)]^{2m+1} \times$$

$$\times [1 + \{(2m+1) - (m^2 + m + 1)\sin z\}/8k\cos^2 z] \pm$$

$$\pm e^{-v_2}[\sin(\tfrac{1}{2}z + \tfrac{1}{4}\pi)]^{2m+1} \times$$

$$\times [1 + \{(2m+1) + (m^2 + m + 1)\sin z\}/8k\cos^2 z]\Big\}\cos^{-m-1} z. \qquad (1, 2)$$

When $\frac{1}{2}\pi < z < \frac{3}{2}\pi$, alter the centre signs to $\mp$. The multipliers are

$$C_{2n} = 2^{2n-\frac{1}{2}} p'_{2n}(\pi k)^{-\frac{1}{2}}, \qquad C_{2n+1} = 2^{2n+\frac{1}{2}} p'_{2n+1}(\pi k)^{-\frac{1}{2}}, \qquad (3)$$

$$S_{2n+1} = 2^{2n-\frac{1}{2}} s'_{2n+1}(\pi k)^{-\frac{1}{2}}, \qquad S_{2n+2} = 2^{2n+\frac{1}{2}} s'_{2n+2}(\pi k)^{-\frac{1}{2}}. \qquad (4)$$

For z real, q large and *positive*

$$\begin{matrix} Ce_m \\ Se_{m+1} \end{matrix}(z, q) \sim \begin{matrix} C_m \\ S_{m+1} \end{matrix} \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} [2k\sinh z - (2m+1)\tan^{-1}(\tanh \tfrac{1}{2}z)]\Big\}$$

$$2^{-m+\frac{1}{2}} \cosh^{-\frac{1}{2}} z. \qquad (5, 6)$$

These results were obtained by uniting zi for z in (1), (2) and retaining the leading terms only. Retention of the neglected terms gives a more accurate result.

The large values of $q = k^2$ for which these functions vanish may be found by aid of (5), (6). Thus for $Ce_{2n}(z, q)$ we must have

$$2k \sinh z \simeq (2r + 1)\tfrac{1}{2}\pi + (4n + 1) \tan^{-1} (\tanh \tfrac{1}{2}z). \tag{7}$$

If e is the eccentricity of the fundamental ellipse, then $\cosh z = e^{-1}$, $\sinh z = (1 - e^2)^{\frac{1}{2}}e^{-1}$, $\tanh \tfrac{1}{2}z = [(1 - e)/(1 + e)]^{\frac{1}{2}} = \theta$. Thus we obtain

$$Ce_{2n}(z, q): \qquad k^2 = q = [(r + \tfrac{1}{2})\pi + (4n + 1) \tan^{-1} \theta]^2 e^2/4(1 - e^2). \tag{8}$$

Similarly we find that

$$Ce_{2n+1}(z, q): \qquad \text{As at (8) but with } (4n + 3) \text{ for } (4n + 1). \tag{9}$$

For $Se_{2n+1}(z, q)$, $Se_{2n+2}(z, q)$ use 1 for $\frac{1}{2}$ in (8) and (9), respectively. In these formulae $r = 1, 2, 3, \cdots$, and the accuracy improves with increase in r, and also as $e \to 1$, when the fundamental ellipse becomes long and narrow.

q large and negative. The formulae for this case are derived (a) by applying relationships (1), (5), (10), (15) § 4 to (1), (2), and altering the range of z accordingly; (b) by applying (1)–(4) § 12 to (5), (6) and selecting the real part where z is real. If z, real, is large enough, $(2m + 1) \tan^{-1} [\tanh (\frac{1}{4}\pi i + \frac{1}{2}z)] \simeq (2m + 1)\frac{1}{4}\pi$. Then the transformed versions of (5), (6) reproduce (13)–(16) §28.

31. Classification of Mathieu functions for a, q real $\leqslant 0$. In tables 1–3, $m = 2n$ or $2n + 1$, $n = 0, 1, 2, \cdots$; $0 < \beta < 1$; μ real and positive. The functions of complex order defined in § 20 have been omitted, because those given in table 3 are preferable for the reasons stated in § 20 in connection with the odd and even solutions. The *cer*, *cei*, etc., functions have been omitted, since q is negative imaginary. The corresponding functions of the second kind derivable from fe_m and ge_m, would be designated fer_m, fei_m, ger_m, gei_m.

Fey, *Gey*, *Fek*, *Gek* have priority over Fe_m, Ge_m, as modified functions of the second kind because:

1. They are better suited for applications, their asymptotic formulae being simpler;
2. Their representations in Bessel function product series converge rapidly, thereby facilitating computation;
3. They degenerate to the Y- and K-Bessel functions when the fundamental ellipse tends to a circle.

The following illustrates the manner in which the functions may be described: $Fey_m(z, q)$ is a Y-type modified Mathieu function of the second kind of order m, q positive, whereas $Fek_m(z, -q)$ is a K-type modified Mathieu function of the second kind of order m, q negative.

For given (a, q), any two corresponding solutions of the first and second kinds of equal order constitute a fundamental system of solutions. For instance the pairs $Ce_{2n+1}(z, -q)$, $Fek_{2n+1}(z, -q)$; $Ce_{2n+1}(z, -q)$, $Fe_{2n+1}(z, -q)$;

$Me_m^{(1)}(z, q)$, $Me_m^{(2)}(z, q)$; $Ne_m^{(1)}(z, -q)$, $Ne_m^{(2)}(z, -q)$, constitute fundamental systems. If y_1 and y_2 are the two solutions of a fundamental system, a complete solution is given by

$$y = Ay_1 + By_2, \tag{1}$$

where A, B are arbitrary constants, determinable in applications by the usual procedure.

TABLE 1

Stable solutions of (1) §1: ordinary functions

Order	First kind	Second kind	Alternative Second kind
Integral	$ce_m(z, \pm q)$	$fe_m(z, \pm q)$	$fek_m(z, -q)$, z real
Integral	$se_m(z, \pm q)$	$ge_m(z, \pm q)$	$gek_m(z, -q)$, z real
Fractional	$ce_{m+\beta}(z, \pm q)$	$se_{m+\beta}(z, \pm q)$	

TABLE 2

Solutions of (3) §1: modified functions

Order	First kind	Second kind	Alternative Second kind	Third kind
Integral	$Ce_m(z, \pm q)$	$Fey_m(z, \pm q)$: $Fek_m(z, \pm q)$	$Fe_m(z, \pm q)$	$Me_m^{(1),\,(3)}(z, \pm q)$
Integral	$Se_m(z, \pm q)$	$Gey_m(z, \pm q)$: $Gek_m(z, \pm q)$	$Ge_m(z, \pm q)$	$Ne_m^{(1),\,(3)}(z, \pm q)$
Fractional	$Ce_{m+\beta}(z, \pm q)$	$Se_{m+\beta}(z, \pm q)$		

TABLE 3

Unstable solutions of (1) §1: functions of order $(m + \mu)$

First kind	Second kind
$ceu_{m+\mu}(z, \pm q)$	$ceu_{m+\mu}(-z, \pm q)$

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APPENDIX

After the paper had been submitted to the Editor, correspondence passed between G. Blanch, W. G. Bickley, and the Author, regarding the definitions of $Fey_m(z, q)$, $Fek_m(z, q)$, and kindred functions given in reference 1. It is agreed that the representations in Bessel function products are the best from every point of view, e.g. analytically, computationally. The series with arguments $2k \sinh z$, $2k \cosh z$ were useful before the others materialised, but they may *now* be regarded as analytical curiosities! Such is the march of progress.

It is opportune to make additional comment on another point mentioned in the correspondence, namely, the relationship between the first and second solutions of (3) § 1, as given in § 9. In the B.F. product series (4) § 8, replace the Y function by its expansion (reference 8, p. 161). Then we get

$$Fey_{2n}(z, q) = \left[\frac{2}{\pi}(\gamma + \log_e \tfrac{1}{2}k)Ce_{2n}(z, q) + (p_{2n}/A_0)\Sigma \cosh \text{ series}\right] + \left[\frac{2}{\pi} z\, Ce_{2n}(z, q) + (p_{2n}/A_0)\Sigma \sinh \text{ series}\right], \tag{1}$$

$$= \text{Even member} + \text{odd member}. \tag{2}$$

Now the differential equation (3) § 1 is such that if one solution be even, a linearly independent one is odd. Accordingly the even member in (1) is a constant multiple of the even solution of the first kind $Ce_{2n}(z, q)$, while the odd member is a constant multiple of the odd solution of the second kind $Fe_{2n}(z, q)$. Thus (1) has the same form as (1), (2) § 9, so by comparison, we have

$$B = Fey'_{2n}(0, q)/Fe'_{2n}(0, q) = 2/\pi C_{2n}(q). \tag{3}$$

Similarly we find that

$$Fey'_{2n+1}(0,q)/Fe'_{2n+1}(0,q) = 2/\pi C_{2n+1}(q), \text{ and}$$

$$Gey_m(0,q)/Ge_m(0,q) = -2/\pi S_m(q). \tag{4}$$

Being the sum of odd and even solutions, $Fey_m(z, q)$, $Gey_m(z, q)$ are neither odd nor even. So far as the infinite integral representations in §§ 8, 10, 15 are concerned, this point is covered by the restrictions $\cosh z > 1$ in w_1, $\sinh z > 0$ in w_2; k, z, real > 0.

In discussing reference 1, G. Blanch pointed out that

$$Fey_{2n}(z + ip\pi, q) = Fey_{2n}(z, q) + 2ip\, Ce_{2n}(z, q), \tag{5}$$

and

$$Fey_{2n+1}(z + 2ip\pi, q) = Fey_{2n+1}(z, q) + 4ip\, Ce_{2n+1}(z, q), \tag{6}$$

p being an integer. Similar relationships exist for $Gey_m(z, q)$, $Fek_m(z, q)$, etc. (5), (6) illustrate that the functions are non-periodic, although this is evident from the odd member of (1). The term $\log_e(ke^z) = (z + \log_e k)$ in the expansion of the Y function is responsible for this feature, which is absent from the infinite integral representations. The restriction on z mentioned above covers this latter point.

Origin of (2), (4), (6), (8) § 8. Writing $-iz^*$ for z in (1) yields

$$\begin{aligned} Fey_{2n}(-iz, q) = &\left[\frac{2}{\pi}(\gamma + \log_e \tfrac{1}{2}k)ce_{2n}(z, q) + (p_{2n}/A_0)\,\Sigma \text{ cos series}\right] \\ &- i\left[\frac{2}{\pi}z\, ce_{2n}(z,\ q) + (p_{2n}/A_0)\Sigma \text{ sin series}\right]. \end{aligned} \tag{7}$$

Comparison with (1) § 3 shows that the odd member in (7) is a constant multiple of $fe_{2n}(z, q)$. We now deduce that

$$C'_{2n} = -\tfrac{1}{2}\pi C_{2n}(q). \tag{8}$$

Similarly we find that

$$C'_{2n+1} = -\tfrac{1}{2}\pi C_{2n+1}(q), \text{ and } S'_m = -\tfrac{1}{2}\pi S_m(q). \tag{9}$$

If desired we could define (7) to be

$$fey_{2n}(z, q) = Fey_{2n}(-iz, q), \tag{10}$$

and also

$$gey_{2n+2}(z, q) = Gey_{2n+2}(-iz, q); \tag{11}$$

but it is doubtful whether a 'complex' second solution would be useful.

By aid of (8), (9) the following relationships may be derived from § 3:

$$p_{2n}l_{2n} = ce_{2n}(0, q) + \Sigma(2r + 2)f_{2r+2}^{(2n)} = fe'_{2n}(0, q)/c_{2n}(q), \tag{12}$$

$$p_{2n+1}l_{2n+1} = ce_{2n+1}(0, q) + \Sigma(2r + 1)f_{2r+1}^{(2n+1)} = fe'_{2n+1}(0, q)/c_{2n+1}(q), \tag{13}$$

$$s_{2n+1}m_{2n+1} = \Sigma\, g_{2r+1}^{(2n+1)} = ge_{2n+1}(0, q)/s_{2n+1}(q), \tag{14}$$

$$-s_{2n+2}m_{2n+2} = \Sigma\, g_{2r}^{(2n+2)} = +ge_{2n+2}(0, q)/s_{2n+2}(q). \tag{15}$$

* The use of $-i$ instead of $+i$ is purely conventional, but is based upon the following: By (1) §§ 7, 9, we get

$$Fey_{2n}(\mp iz, q) = Ace_{2n}(z, q) - i\, Bfe_{2n}(\pm z, q),$$

so that $-iz$ in Fey_{2n} corresponds to $+z$ in fe_{2n}.

Approximate formulae for Fey_m, Gey_m, Fek_m, Gek_m, q small and positive.

If q and $ke^{\pm z}$ are small enough, the B. F. product series may be used to derive the forms of these functions near $q = +0$. Under this condition, the coefficients $A_p^{(m)}$, $B_p^{(m)}$ in the series are given by [7a]:

$$c_{m+2r}^{(m)} \cong (-1)^r m!\, q^r/2^{2r}\, r!(m+r)! \qquad (16)$$

$$(r \geqq 0, m > 0).$$

$$c_{m-2r}^{(m)} \cong (m-r-1)!\, q^r/2^{2r}(m-1)! \qquad (17)$$

Using (16), (17) in the B. F. series, we derive the first approximations:

$$Fey_0(z, q) \simeq 2^{1/2}\pi^{-1}(z + \log_e k), \qquad (18)$$

$$\begin{matrix} Fey \\ Gey_m \end{matrix}(z, q) \cong -2^{2m-1}\pi^{-1}(m-1)!\, m!\, e^{-mz} q^{-m} \qquad (m \geqq 1) \qquad (19)$$

$$\begin{matrix} Fek \\ Gek_m \end{matrix}(z, -q) \cong -\tfrac{1}{2} \begin{matrix} Fey \\ Gey_m \end{matrix}(z, q) \qquad (m \geqq 0) \qquad (20)$$

These formulae were obtained independently by G. Blanch and the Author.

Singularities. In (18)–(20) the functions have singularities at $q = 0$. Fey_0, Fek_0 have branch points due to $\log_e k$, while for $m \geqq 1$, the functions have poles of order m in q. The multipliers outside the $\sum$ in the B. F. product series have singularities, which may be ascertained by aid of (16), (17). For example, as $q \to +0$.

$$(p_{2n}/A_0) \to (-1)^n 2^{4n-2}[(2n)!]^2 q^{-2n}, \qquad (n \geqq 1) \qquad (21)$$

$$(p_{2n+1}/A_1) \to (-1)^n 2^{4n}(2n+1)[(2n)!]^2 q^{-2n-1/2}. \qquad (\geqq 0) \qquad (22)$$

(21) has a pole of order $2n$, while (22) has a branch point at $q = 0$. Hence the B. F. series (*alone*) for $Fey_{2n} \cdots$ have no singularities. Functions of order $(2n+1)$ have a branch point at $q = 0$ due to the factor $q^{-1/2}$.

When $z = 0$, formulae more accurate than (18)–(20) may be derived from results given in reference 1*b*, provided that allowance is made for the difference in the definitions of the functions, and the forms of the differential equations in the two cases.

LONDON, ENGLAND.

See correction
V, 2c - no -1, q - 78

NOTE ON THE SHEAR STRESSES IN A BENT CANTILEVER BEAM OF RECTANGULAR CROSS SECTION

By E. Reissner and G. B. Thomas

In this note we wish to report on some calculations which we have made concerning the distribution of shear stresses, according to the St. Venant theory, in bent cantilevers of rectangular cross section.[1] Referring to figure 1, our main result is the fact that for sufficiently wide beams ($b/a \gg 1$) the component of stress $\tau_{ys}(a, \eta)$ not only is of the same order of magnitude as the stress $\tau_{zz}(0, b)$ for which values have previously been calculated[1], but exceeds $\tau_{zz}(0, b)$ in magnitude.[2]

We list first the exact expressions for $\tau_{zz}(x, y)$ and $\tau_{ys}(x, y)$ in a form convenient for our purposes and deduce from them simplified expressions for $\tau_{zz}(0, b)$ and $\tau_{ys}(a, y)$ which ensure accuracy for all decimals computed, when $b/a \geq 4$. On the basis of these formulas we calculate for a number of values of b/a in the range $(2, \infty)$ (i) values of $\tau_{zz}(0, b)$, (ii) values of the coordinate $y = \eta$ for which $\tau_{ys}(a, y)$ is greatest, (iii) values of $\tau_{ys}(a, \eta)$. The numerical results obtained are collected in Table I.

Exact expressions for the shear stresses may be written in the following form, with $4ab = A$,

$$\tau_{zz} = \frac{3P}{2A}\left\{1 - \frac{x^2}{a^2} + \frac{\nu}{1+\nu}\frac{32}{\pi^3}\sum_{n=0}^{\infty}\frac{(-1)^n \cos\frac{(2n+1)\pi}{2}\frac{x}{a}}{(2n+1)^3} \times\left[\frac{(2n+1)\pi}{2}\frac{b}{a}\frac{\cosh\frac{(2n+1)\pi}{2}\frac{y}{a}}{\sinh\frac{(2n+1)\pi}{2}\frac{b}{a}} - 1\right]\right\} \tag{1}$$

$$\tau_{ys} = \frac{3P}{2A}\left\{\frac{\nu}{1+\nu}\frac{16b}{\pi^3 a}\sum_{n=0}^{\infty}\frac{(-1)^n \sin\frac{(2n+1)\pi}{2}\frac{x}{a}}{(2n+1)^3} \times\left[\frac{y}{b} - \frac{\sinh\frac{(2n+1)\pi}{2}\frac{y}{a}}{\sinh\frac{(2n+1)\pi}{2}\frac{b}{a}}\right]\right\} \tag{2}$$

[1] For a detailed exposition of the theory see S. Timoshenko's book on Theory of Elasticity (pp. 285–288, 292–298). We recall that in this theory the manner in which the load P at the free end is applied cannot be arbitrarily specified, but must be taken in accordance with Eqs. (1) and (2), and likewise the condition of *complete* restraint at the fixed end cannot be satisfied. According to St. Venant's principle this has negligible effect on the stress distribution at distances away from the ends of the beam which are of the order of magnitude of the width $2b$ (assuming that $a < b$).

[2] We are indebted to Prof. S. Timoshenko and to Prof. H. Reissner for the information that this fact appears not to have been noted previously.

From equations (1) and (2) are derived the following approximate expressions which are exact within the accuracy of our calculations when $b/a \geq 4$:

$$\tau_{xz}(0, b) = \frac{3P}{2A}\left\{1 + \frac{\nu}{1+\nu}\frac{32}{\pi^3}\sum_0^\infty \frac{(-1)^n\left[\frac{(2n+1)\pi}{2}\frac{b}{a} - 1\right]}{(2n+1)^3}\right\} \tag{3}$$

$$\tau_{yz}(a, y) = \frac{3P}{2A}\left\{\frac{\nu}{1+\nu}\frac{16b}{\pi^3 a}\sum_0^\infty \frac{\frac{y}{b} - \exp\left[\frac{(2n+1)\pi}{2}\frac{y-b}{a}\right]}{(2n+1)^3}\right\} \tag{4}$$

Substituting in equation (3) the series sums

$$\sum \frac{(-1)^n}{(2n+1)^2} = \frac{\pi^2}{32}, \qquad \sum \frac{(-1)^n}{(2n+1)^3} = 0.9160 \text{ (Catalan's constant)}$$

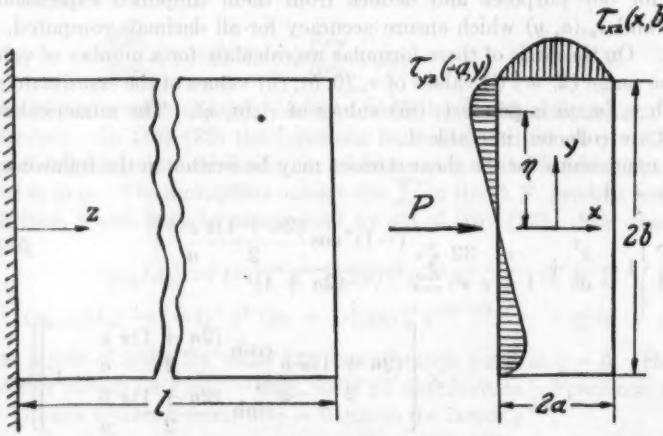

FIG. 1. DIAGRAM SHOWING DIMENSIONS OF CANTILEVER BEAM AND NATURE OF SHEAR STRESS DISTRIBUTION

this equation becomes

$$\tau_{xz}(0, b) = \frac{3P}{2A}\left\{1 + \frac{\nu}{1+\nu}\left[1.4849\frac{b}{a} - 1\right]\right\} \tag{5}$$

To obtain the maximum of $\tau_{yz}(a, y)$ we set $(\partial\tau_{yz}(a, y)/\partial y)_{y=\eta} = 0$. According to equation (4) the maximum condition becomes

$$\sum_{n=0}^\infty \frac{1}{(2n+1)^2} - \frac{\pi b}{2a}\sum_{n=0}^\infty \frac{\exp\left[\frac{(2n+1)\pi}{2}\frac{\eta - b}{a}\right]}{2n+1} = 0 \tag{6}$$

Carrying out the summation in (6) we have

$$\frac{\pi^2}{8} - \frac{\pi b}{2a}\tanh^{-1}(e^{\pi(\eta-b)/2a}) = 0 \tag{7}$$

or

$$\frac{\pi}{2}\frac{(\eta - b)}{a} = \ln\left(\tanh\frac{\pi a}{4b}\right) \tag{8a}$$

Equation (8a) may also be written in a form which gives directly the distance of the maximum location from the edge in units of the thickness $2a$,

$$\frac{b - \eta}{2a} = \frac{1}{\pi}\ln\left(\coth\frac{\pi a}{4b}\right) \approx \frac{1}{\pi}\ln\frac{4b}{\pi a} \tag{8b}$$

Introducing equation (8a) in equation (4) we obtain

TABLE I

$\frac{b}{a}$	$\frac{\tau_{xx}(0, b)}{3P/2A}$	$\frac{\tau_{yx}(a, \eta)}{3P/2A}$	$\frac{\tau_{yx}(a, \eta)}{\tau_{xx}(0, b)}$	$\frac{b - \eta}{2a}$
0	1.000	0.000	0.000	0.000
2	1.39(4)	0.31(6)	0.22(7)	0.31(4)
4	1.988	0.968	0.487	0.522
6	2.582	1.695	0.656	0.649
8	3.176	2.452	0.772	0.739
10	3.770	3.226	0.856	0.810
15	5.255	5.202	0.990	0.939
20	6.740	7.209	1.070	1.030
25	8.225	9.233	1.123	1.102
50	15.650	19.466	1.244	1.322
∞	∞	∞	1.347	∞

$$\tau_{yx}(a, \eta) = \frac{3P}{2A}\left\{\frac{\nu}{1+\nu}\frac{16b}{\pi^2 a}\left[\frac{\pi^2}{8} + \frac{\pi a}{4b}\ln\left(\tanh\frac{\pi a}{4b}\right) - \sum\frac{1}{(2n+1)^2}\left(\tanh\frac{\pi a}{4b}\right)^{2n+1}\right]\right\} \tag{9}$$

Equation (9) is less simple in appearance than equation (5) but is readily evaluated as the remaining series converges rapidly.

From (9) and (5) we obtain the following limit relation

$$\lim_{b/a\to\infty}\frac{\tau_{yx}(a, \eta)}{\tau_{xx}(0, b)} = \frac{2}{1.4849} = 1.347, \tag{10}$$

which shows that for wide beams (plates) the component of shear parallel to the face of the plate reaches a value which is almost 35 percent higher than the value reached by the component of transverse shear. The way in which this limit state is approached is apparent from the values given in Table I which has been computed on the basis of equations (5), (8b), and (9). In the calculations the value of ν has been taken as .25.

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COEFFICIENTS FOR FACILITATING THE USE OF THE GAUSSIAN QUADRATURE FORMULA

By Herbert E. Salzer*

The well known Gaussian quadrature formula which is usually employed in the form

$$\int_{-1}^{1} f(x)\,dx \sim \sum_{i=1}^{n} a_i f(x_i),$$

where x_i denotes the i^{th} zero of the Legendre polynomial $P_n(x)$ and $a_i = \frac{1}{P_n'(x_i)} \int_{-1}^{1} \frac{P_n(x)}{x - x_i}\,dx$, uses only n points, but is exact when $f(x)$ is any polynomial of degree $2n - 1$. (If the interval of integration is $[a, b]$, it is transformed into $[-1, 1]$ by the change of variable $x = \frac{b-a}{2} y + \frac{b+a}{2}$). The Gaussian formula suffers from the inconvenience of requiring the function at irregularly separated points. Apart from the problem of finding the value of the function at irregularly separated points, there is the added difficulty of checking those values, since one cannot perform an ordinary differencing test.

But it is possible to obtain some idea of the accuracy by taking the divided differences,** which correspond in magnitude not to the differences, but to the reduced derivatives $f^{(m)}(x)/m!$, so that if we make use of the approximate relationship, mth difference $\sim h^m \times m$th derivative, where h = tabular interval, what would correspond to the mth difference in the case of an irregularly spaced function would be the mth divided difference $\times\, m! \times$ some "average value of h^m." To judge the correctness most effectively by a differencing or divided difference test, one must see all the differences. But knowledge that the $(n - 1)$th difference of n values is small, without knowledge of all the other differences, usually affords some indication about the correctness of the functions, according to the technique of investigating the propagation of error in the differencing process, which is so well known to computers. However one must bear in mind that a single small $(n - 1)$th difference gives no guarantee of the magnitude of error, because it might just happen to be at the place where the differences are still large but changing in sign, or that particular value of the $(n - 1)$th difference might just happen to be small, even when there is a gross error in the function.

Now it is a rather tedious process to obtain the $(n - 1)$th divided difference from n values, say for $n = 10$; moreover it is very easy to make a simple error

* Mathematical Tables Project, National Bureau of Standards.

** For a discussion of the definition, properties and uses of divided differences, one may consult L. M. Milne-Thomson, *Calculus of Finite Differences*, Chapter I, pp. 1–19, E. T. Whittaker and G. Robinson, *Calculus of Observations*, (4th ed.), pp. 20–28, or J. F.S teffensen, *Interpolation*, pp. 14–26.

that throws off the entire computation, because in taking divided differences there is no easy way of checking all the divided differences of a particular order before proceeding to the next order, as we can do when differencing functions tabulated at equal intervals. However, the divided difference of any order of $f(x_i)$ is a linear function of entries, with coefficients that are functions of the quantities x_i. In particular the $(n-1)$th divided difference of $f(x_i)$, $i = 1, \cdots, n$, is equal to

$$\begin{vmatrix} f(x_1) & f(x_2) & \cdots & f(x_n) \\ x_1^{n-2} & x_2^{n-2} & \cdots & x_n^{n-2} \\ x_1^{n-3} & x_2^{n-3} & \cdots & x_n^{n-3} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ x_1 & x_2 & \cdots & x_n \\ 1 & 1 & \cdots & 1 \end{vmatrix} \div \begin{vmatrix} x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \\ x_1^{n-2} & x_2^{n-2} & \cdots & x_n^{n-2} \\ x_1^{n-3} & x_2^{n-3} & \cdots & x_n^{n-3} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ x_1 & x_2 & \cdots & x_n \\ 1 & 1 & \cdots & 1 \end{vmatrix}, \quad \text{or,}$$

more simply, to $\sum_{i=1}^{n} f(x_i)/$difference-product of x_i with $x_0, x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n$. Since the Gaussian formula employs the same set of x_i's, for each n, the $(n-1)$th divided difference can be expressed as $\sum_{i=1}^{n} C_i^n f(x_i)$, where the quantities C_i^n can be found once and for all, to be used with any $f(x)$.

The table below gives the quantities C_i^n, for $n = 3, 4, \cdots, 10$; for $n = 3, 4$ and 5, to $8D$; for $n = 6$, 7, and 8, to $7D$; for $n = 9$ and 10, to $6D$. They are correct to within about a unit in the last decimal place which is given. They were calculated from the values of x_i given in the paper by A. N. Lowan, N. Davids, A. Levenson, "Table of the Zeros of the Legendre Polynomials of Order 1–16 and the Weight Coefficients for Gauss' Mechanical Quadrature Formula," Bull. Amer. Math. Soc., vol. 48, No. 10, pp. 739–743, Oct. 1942. (Their notation for x_i employs positive and negative subscripts.) For n odd, $C_i^n = C_{n-i+1}^n$, and for n even, $C_i^n = -C_{n-i+1}^n$. These relations were used as a check upon the quantities C_i^n, all of which were computed independently. The middle coefficient for odd n was the only one that could not be checked that way. As a further check upon the quantities C_i^n, they were used to calculate the divided differences of x_i for n even, and of x_i^2 for n odd and > 3, which, of course, must equal zero.

To obtain an idea of the convenience of these coefficients, suppose that it is desired to check the integrand in the ten-point Gaussian quadrature formula by finding the 9th divided difference. To do so the quickest method which does not use auxiliary quantities would require at least 90 subtractions and 45 divisions, and it would be necessary to write down all the intermediate divided differences (44 quantities) and the whole process would be complicated by the need for taking constant account of the significant figures and the propagation of error inherent in so many arithmetic operations. On the other hand, with the coefficients C_i^{10}, the answer is obtainable almost immediately by performing

only 10 multiplications, cumulating all the products in the machine, so that no writing is necessary.

Another delicate point about the value of this divided difference check, apart from the inconclusiveness of a single difference, is that when the Gaussian formula is used " up to the hilt," i.e. where one actually requires the full $2n - 1$ degree accuracy from n points, the very fact that it is being used, precludes any possibility of a completely adequate differencing check. The reason is that if it were possible to employ a differencing check as far as the last digits, then there would have been no need for the Gaussian formula in the first place, since a formula of the Cotes type would have been adequate. Thus, in conclusion, it is to be remembered that these coefficients serve merely as a partial check that will not take care of the last few digits when the Gaussian formula is employed with its maximum efficiency, and in any case they are to be used to indicate only "probable correctness" of the quantities $f(x_i)$.

TABLE OF COEFFICIENTS C_i^n

$n = 3$

$C_1^3 = C_3^3 = 0.83333\ 333$

$C_2^3 = -1.66666\ 667$

$n = 4$

$C_1^4 = -C_4^4 = -0.92756\ 751$

$C_2^4 = -C_3^4 = 2.34943\ 118$

$n = 5$

$C_1^5 = C_5^5 = 1.14623\ 257$

$C_2^5 = C_4^5 = -3.24623\ 257$

$C_3^5 = 4.20000\ 000$

$n = 6$

$C_1^6 = -C_6^6 = -1.52648\ 66$

$C_2^6 = -C_5^6 = 4.60007\ 60$

$C_3^6 = -C_4^6 = -6.78156\ 38$

$n = 7$

$C_1^7 = C_7^7 = 2.14869\ 64$

$C_2^7 = C_6^7 = -6.72732\ 54$

$C_3^7 = C_5^7 = 10.70720\ 04$

$C_4^7 = -12.25714\ 29$

$n = 8$

$C_1^8 = -C_8^8 = -3.15562\ 90$

$C_2^8 = -C_7^8 = 10.13236\ 16$

$C_3^8 = -C_6^8 = -16.93945\ 52$

$C_4^8 = -C_5^8 = 21.04530\ 71$

$n = 9$

$C_1^9 = C_9^9 = 4.79205\ 7$

$C_2^9 = C_8^9 = -15.65912\ 6$

$C_3^9 = C_7^9 = 27.07324\ 2$

$C_4^9 = C_6^9 = -35.49982\ 4$

$C_5^9 = 38.58730\ 2$

$n = 10$

$C_1^{10} = -C_{10}^{10} = -7.47622\ 6$

$C_2^{10} = -C_9^{10} = 24.74259\ 5$

$C_3^{10} = -C_8^{10} = -43.81722\ 4$

$C_4^{10} = -C_7^{10} = 59.66202\ 5$

$C_5^{10} = -C_6^{10} = -68.58249\ 0$

SIMILARITY LAWS OF HYPERSONIC FLOWS

By Hsue-shen Tsien

Introduction. Hypersonic flows are flow fields where the fluid velocity is much larger than the velocity of propagation of small disturbances, the velocity of sound. Th. von Kármán [1] has pointed out that in many ways the dynamics of hypersonic flows is similar to Newton's corpuscular theory of aerodynamics. The pressure acting on an inclined surface is thus greater than the free stream pressure by a quantity which is approximately proportional to the square of the angle of inclination instead of the usual linear law for conventional supersonic flows. E. Sänger [2] has, in fact, used this concept to design the optimum wing and body shapes for hypersonic flight at extreme speeds.

Recently, von Kármán [3] has obtained the similarity laws for transonic flows where the fluid velocity is very near to the velocity of sound. He deduced these laws by using an affine transformation of the fluid field so that the differential equations of the flows are reduced to a single non-dimensional equation. In this paper, the same method is used to derive the similarity laws for hypersonic flows. These laws will be, perhaps, useful in correlating the experimental data to be obtained in the near future by hypersonic wind tunnels now under construction.

Differential Equation for Hypersonic Flows. If u, v are the components of velocity in the x, y directions and a is the local velocity of sound, the differential equations for irrotational two-dimensional motion are

$$\left(1 - \frac{u^2}{a^2}\right)\frac{\partial u}{\partial x} - \frac{uv}{a^2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \left(1 - \frac{v^2}{a^2}\right)\frac{\partial v}{\partial y} = 0 \tag{1}$$

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \tag{2}$$

Now if a slender body is present in an otherwise uniform stream of velocity V in the x-direction, equation (2) is satisfied by introducing the disturbance velocity potential φ defined as

$$u = V + \frac{\partial \varphi}{\partial x}, \qquad v = \frac{\partial \varphi}{\partial y}. \tag{3}$$

If a_0 is the velocity of sound for the gas at rest and a^0 is the velocity of sound corresponding to the free stream velocity V, then there are the following relations:

$$\left.\begin{aligned} a^2 &= a_0^2 - \frac{\gamma - 1}{2}(u^2 + v^2) = a_0^2 \\ &\quad - \frac{\gamma - 1}{2}\left[V^2 + 2V\frac{\partial \varphi}{\partial x} + \left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2\right] \\ a^{0^2} &= a_0^2 - \frac{\gamma - 1}{2}V^2. \end{aligned}\right\} \tag{4}$$

where γ is the ratio of the specific heats.

For hypersonic flows over a slender body, both a^0 and $\frac{\partial\varphi}{\partial x}$, $\frac{\partial\varphi}{\partial y}$ are small in comparison with V. By substituting equations (3) and (4) into equation (1) and retaining terms up to second order, one has

$$\left[1-(\gamma+1)M^0\frac{1}{a^0}\frac{\partial\varphi}{\partial x}-\frac{\gamma-1}{2}\frac{1}{a^{0^2}}\left(\frac{\partial\varphi}{\partial y}\right)^2-M^{0^2}\right]\frac{\partial^2\varphi}{\partial x^2}-2M^0\frac{1}{a^0}\frac{\partial\varphi}{\partial y}\frac{\partial^2\varphi}{\partial x\,\partial y}$$
$$+\left[1-(\gamma-1)M^0\frac{1}{a^0}\frac{\partial\varphi}{\partial x}-\frac{\gamma+1}{2}\frac{1}{a^{0^2}}\left(\frac{\partial\varphi}{\partial y}\right)^2\right]\frac{\partial^2\varphi}{\partial y^2}=0. \tag{5}$$

Here M^0 is the Mach number of the free stream or

$$M^0=\frac{V}{a^0}. \tag{6}$$

Similarity Laws in Two-Dimensional Flow. If $2b$ is the length or chord of the body and δ the thickness of the body, the non-dimensional coordinates ξ and η can be defined as

$$x=b\xi, \qquad y=b\left(\frac{\delta}{b}\right)^n\eta \tag{7}$$

where n is the exponent yet to be determined. von Kármán [1] has shown that for hypersonic flow over a slender body the variation of fluid velocity due to the presence of the body is limited within a narrow region close to the body, the hypersonic boundary layer. Therefore, in order to investigate this velocity variation, one must expand the coordinate normal to the surface of the body. This is similar to the case of ordinary viscous boundary layer, where Prandtl's simplified boundary layer equation is obtained from the exact Navier-Stokes equations by a coordinate expansion normal to the surface of the body. From this reasoning then, n must be positive so that η is much greater than $(y)/(b)$. This surmise is substantiated by the later calculations to be shown presently.

The appropriate non-dimensional form for the velocity potential φ is

$$\varphi=a^0b\frac{1}{M^0}f(\xi,\eta). \tag{8}$$

By substituting equations (7) and (8) into equation (5), one has

$$\left(\frac{\delta}{b}\right)^{2n}\left[1-(\gamma+1)\frac{\partial f}{\partial\xi}-\frac{\gamma-1}{2}\frac{1}{M^{0^2}(\delta/b)^{2n}}\left(\frac{\partial f}{\partial\eta}\right)^2\right]\frac{\partial^2 f}{\partial\xi^2}$$
$$-M^{0^2}\left(\frac{\delta}{b}\right)^{2n}\frac{\partial^2 f}{\partial\xi^2}-2\frac{\partial f}{\partial\eta}\frac{\partial^2 f}{\partial\xi\,\partial\eta} \tag{9}$$
$$+\left[1-(\gamma-1)\frac{\partial f}{\partial\xi}-\frac{\gamma+1}{2}\frac{1}{M^{0^2}(\delta/b)^{2n}}\left(\frac{\partial f}{\partial\eta}\right)^2\right]\frac{\partial^2 f}{\partial\eta^2}=0.$$

The boundary conditions at infinity require that the flow velocity be V. Thus

$$\frac{\partial f}{\partial\xi}=\frac{\partial f}{\partial\eta}=0 \quad \text{at} \quad \infty \tag{10}$$

If the slender body is a symmetrical one, then the condition at the surface of the body can be written as

$$\left(\frac{\partial \varphi}{\partial y}\right)_{y=0} = a^0 M^0 \left(\frac{\delta}{b}\right) h(\xi) \quad \text{for} \quad -1 < \xi < 1 \tag{11}$$

where $h(\xi)$ for $-1 < \xi < 1$ is a given function describing the thickness distribution along the length of the body. Equation (11) can be converted into the following form by means of equations (7) and (8):

$$\left(\frac{\partial f}{\partial \eta}\right)_{\eta=0} = M^{0^2} \left(\frac{\delta}{b}\right)^{1+n} h(\xi) \tag{12}$$

Since the body is thin, δ/b is very small. Therefore the first group of terms in equation (9) is negligible in comparison with the rest. Then both the differential equation and the boundary conditions can be made to contain only a single parameter if one sets

$$n = 1 \tag{13}$$

That is, if

$$M^0 \frac{\delta}{b} = K$$

Then equation (9) becomes

$$\left[1 - (\gamma - 1)\frac{\partial f}{\partial \xi} - \frac{\gamma + 1}{2}\frac{1}{K^2}\left(\frac{\partial f}{\partial \eta}\right)^2\right]\frac{\partial^3 f}{\partial \eta^3} = K^2 \frac{\partial^3 f}{\partial \xi^2} + 2\frac{\partial f}{\partial \eta}\frac{\partial^2 f}{\partial \xi\, \partial \eta} \tag{14}$$

and the boundary conditions become

$$\frac{\partial f}{\partial \xi} = \frac{\partial f}{\partial \eta} = 0 \quad \text{at} \quad \infty \tag{15}$$

and

$$\left(\frac{\partial f}{\partial \eta}\right)_{\eta=0} = K^2 h(\xi) \quad \text{for} \quad -1 < \xi < 1.$$

The meaning of this similarity law is the following: If a series of bodies having the same thickness distribution but different thickness ratios (δ/b) are put into flows of different Mach numbers M^0 such that the products of M^0 and (δ/b) remain constant and equal to K, then the flow patterns are similar in the sense that they are governed by the same function $f(\xi, \eta)$, determined by equations (14) and (15).

If p_0 is the stagnation pressure, p^0 the free stream pressure, and p the local pressure, then

$$p^0 = p_0\left[1 + \frac{\gamma - 1}{2} M^{0^2}\right]^{-(\gamma/\gamma-1)}$$

$$p = p_0\left[1 + \frac{\gamma - 1}{2}\frac{u^2 + v^2}{a^2}\right]^{-(\gamma/\gamma-1)}$$

In notations introduced previously and by retaining terms of proper magnitudes, one can write the expression for the local pressure as

$$p = p^0\left[1 - (\gamma - 1)\frac{\partial f}{\partial \xi} - \frac{\gamma - 1}{2}\frac{1}{K^2}\left(\frac{\partial f}{\partial \eta}\right)^2\right]^{\gamma/\gamma-1}. \tag{16}$$

The drag D of the body can then be calculated. It is given by the following expression:

$$D = 2\int_{-b}^{b} (p)_{\eta=0}\, h(\xi)\left(\frac{\delta}{b}\right) dx$$

$$= 2bp^0\left(\frac{\delta}{b}\right)\int_{-1}^{1}\left[1 - (\gamma - 1)\frac{\partial f}{\partial \xi} - \frac{\gamma - 1}{2}\frac{1}{K^2}\left(\frac{\partial f}{\partial \eta}\right)^2\right]_{\eta=0}^{\gamma/\gamma-1} h(\xi)\, d\xi.$$

If one wishes to compute the drag coefficient C_D, then one uses

$$C_D = \frac{D}{\frac{1}{2}\rho^0 V^2(2b)}$$

$$= \frac{1}{M^{0^2}} \cdot \left\{\frac{2}{\gamma} K \int_{-1}^{1}\left[1 - (\gamma - 1)\frac{\partial f}{\partial \xi} - \frac{\gamma - 1}{2}\frac{1}{K^2}\left(\frac{\partial f}{\partial \eta}\right)^2\right]_{\eta=0}^{\gamma/\gamma-1} h(\xi)\, d\xi\right\}. \tag{17}$$

For a given thickness distribution, the quantity within the brackets is only a function of K, the similarity parameter. Therefore, one can write

$$C_D = \frac{1}{M^{0^2}}\Delta(K) = \frac{1}{M^{0^2}}\Delta\left(M^0\frac{\delta}{b}\right). \tag{18}$$

Similarly, one obtains for the lift coefficient C_L of the lift L the following law:

$$C_L = \frac{L}{\frac{1}{2}\rho^0 V^2(2b)} = \frac{1}{M^{0^2}}\Lambda(K) = \frac{1}{M^{0^2}}\Lambda\left(M^0\frac{\delta}{b}\right). \tag{19}$$

These similarity laws show that for bodies of the same thickness distribution at angles of attack proportional to the thickness ratio (δ/b), the quantities $(C_D M^{0^2})$ and $(C_L M^{0^2})$ are functions of the single parameter $K = M^0(\delta/b)$.

Equations (18) and (19) agree with the results of the more limited linearized theory of Ackeret [4]. According to this theory, for similar bodies in the sense stated above the drag coefficient and the lift coefficient are given by

$$C_D \sim \frac{\left(\frac{\delta}{b}\right)^2}{\sqrt{M^{0^2} - 1}}$$

$$C_L \sim \frac{\frac{\delta}{b}}{\sqrt{M^{0^2} - 1}}.$$

For hypersonic flows of very large values of M^0, these expressions reduce to

$$C_D \sim \left(\frac{\delta}{b}\right)^2 \Big/ M^0 \tag{20}$$

$$C_L \sim \left(\frac{\delta}{b}\right) \Big/ M^0. \tag{21}$$

Equations (20) and (21) agree with equations (18) and (19). Equations (18) and (19) are, however, more general and complete.

Axially Symmetrical Flows. For axially symmetrical flows, the ordinate y is the radial distance from the axis to the point concerned. Then a similar analysis leads to the following differential equation and boundary conditions:

$$\left[1 - (\gamma - 1)\frac{\partial f}{\partial \xi} - \frac{\gamma + 1}{2}\frac{1}{K^2}\left(\frac{\partial f}{\partial \eta}\right)^2\right]\frac{\partial^2 f}{\partial \eta^2} + \left[1 - (\gamma - 1)\frac{\partial f}{\partial \xi} - \frac{\gamma - 1}{2}\frac{1}{K^2}\left(\frac{\partial f}{\partial \eta}\right)^2\right]\frac{1}{\eta}\frac{\partial f}{\partial \eta} = 2\frac{\partial f}{\partial \eta}\frac{\partial^2 f}{\partial \xi\, \partial \eta} + K^2\frac{\partial^2 f}{\partial \xi^2} \tag{22}$$

$$\frac{\partial f}{\partial \xi} = \frac{\partial f}{\partial \eta} = 0 \quad \text{at} \quad \infty \tag{23}$$

$$\left(\eta\frac{\partial f}{\partial \eta}\right)_{\eta=0} = K^2 h(\xi) \quad \text{for} \quad -1 < \xi < 1 \tag{24}$$

where $h(\xi)$ is the distribution function for cross-sectional areas along the length of the body and $K = M^0(\delta/b)$ as for two-dimensional flow.

The drag coefficient C_D referred to the maximum cross section of the body is then governed by the following similarity law:

$$C_D = \frac{1}{M^{0^2}}\Delta\left(M^0\frac{\delta}{b}\right). \tag{25}$$

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1. Th. von Kármán, "The Problem of Resistance in Compressible Fluids," Proceedings of the 5th Volta Congress, pp. 275–277, Rome (1936).
2. E. Sänger, "Gleitkörper für sehr hohe Fluggeschwindigkeiten," German Patent 411/42, Berlin (1939).
3. Th. von Kármán, "Similarity Laws of Transonic Flows," to be published soon.
4. See for instance Durand "Aerodynamic Theory," pp. 234–236, Vol. III, J. Springer, Berlin (1935).

TABLE OF THE STRUVE FUNCTIONS $L_\nu(x)$ AND $H_\nu(x)$

By the Mathematical Tables Project

By definition (see G. N. Watson's "A Treatise on the Theory of Bessel Functions," Chapter X)

$$L_\nu(x) = \sum_{m=0}^{\infty} a_m(x); \qquad H_\nu(x) = \sum_{m=0}^{\infty} (-1)^m a_m(x)$$

where

$$a_m(x) = \left(\frac{x}{2}\right)^{2m+1+\nu} \Big/ \Gamma\left(m+\frac{3}{2}\right)\Gamma\left(m+\nu+\frac{3}{2}\right).$$

Thus

$$L_\nu(x) = A_\nu(x) + B_\nu(x)$$
$$H_\nu(x) = A_\nu(x) - B_\nu(x),$$

where

$$A_\nu(x) = \sum_{m=0}^{\infty} a_{2m}(x); \qquad B_\nu(x) = \sum_{m=0}^{\infty} a_{2m+1}(x)$$

The functions $H_\nu(x)$, $L_\nu(x)$ satisfy the relations

(1) $$H_{\nu-1}(x) + H_{\nu+1}(x) = \frac{2\nu}{x} H_\nu(x) + \left[\left(\frac{x}{2}\right)^\nu \Big/ \Gamma\left(\nu+\frac{3}{2}\right)\Gamma(\tfrac{1}{2})\right]$$

(2) $$H_{\nu-1}(x) - H_{\nu+1}(x) = 2H'_\nu(x) - \left[\left(\frac{x}{2}\right)^\nu \Big/ \Gamma\left(\nu+\frac{3}{2}\right)\Gamma(\tfrac{1}{2})\right]$$

(3) $$L_{\nu-1}(x) - L_{\nu+1}(x) = \frac{2\nu}{x} L_\nu(x) + \left[\left(\frac{x}{2}\right)^\nu \Big/ \Gamma\left(\nu+\frac{3}{2}\right)\Gamma(\tfrac{1}{2})\right]$$

(4) $$L_{\nu-1}(x) + L_{\nu+1}(x) = 2L'_\nu(x) - \left[\left(\frac{x}{2}\right)^\nu \Big/ \Gamma\left(\nu+\frac{3}{2}\right)\Gamma(\tfrac{1}{2})\right]$$

(5) $$H_1(x) = \frac{2}{\pi} - H_{-1}(x)$$

(6) $$L_1(x) = -\frac{2}{\pi} + L_{-1}(x)$$

G. N. Watson (loc. cit.) tabulates $H_0(x)$ and $H_1(x)$ from $x = 0$ to $x = 16$ at intervals of 0.02, to 7 decimals, without differences. Linear interpolation in Watson's tables is correct to about 5 decimals; hence his tables are more convenient than ours whenever an accuracy of 5 decimal places is sufficient. If seven-place accuracy is desired, however, our tables might prove more convenient over the overlapping range. The reason our values $H_\nu(x)$ are included

here is because they were obtainable from the computations relating to $L_\nu(x)$ with hardly any additional labor. Furthermore $H_{-2}(x)$ is not given by Watson, and $H_{-1}(x)$ is of course slightly different from $H_1(x)$. In a few cases, the rounding of our values differ by a unit from Watson's. We had no way of determining from our work sheets whether or not our roundings were better, and since the slight variations did not warrant extensive investigation, we are merely calling attention in the table to the discrepancies.

MATHEMATICAL TABLES PROJECT
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x	$L_0(x)$	δ^3	$*\delta^4$	x	$L_0(x)$	δ^3	$*\delta^4$
0.0	0.00000000	00000	000				
.1	.06373274	42526	340	5.1	29.657759	245636	2154
.2	.12789074	85393	683	5.2	32.455237	269664	2369
.3	.19290268	128944	1031	5.3	35.522379	296065	2604
.4	.25920405	173528	1389	5.4	38.885586	325076	2864
.5	.32724070	219503	1758	5.5	42.573870	356955	3149
0.6	0.39747238	267240	2143	5.6	46.619108	391990	3463
.7	.47037645	317123	2546	5.7	51.056337	430493	3808
.8	.54645176	369557	2971	5.8	55.924058	472813	4188
.9	.62622264	424967	3421	5.9	61.264592	519328	4606
1.0	.71024319	483804	3901	6.0	67.124455	570459	5066
1.1	0.79910177	546549	4415	6.1	73.554776	626665	5573
1.2	.89342586	613717	4966	6.2	80.611762	688455	6130
1.3	.99388710	685859	5561	6.3	88.357204	756387	6743
1.4	1.10120695	763572	6203	6.4	96.859032	831074	7418
1.5	1.21616251	847500	6899	6.5	106.19193	91319	816
1.6	1.33959308	938339	7656	6.6	116.43803	100349	898
1.7	1.47240703	1036848	8478	6.7	127.68761	110277	988
1.8	1.61558947	1143850	9374	6.8	140.03996	121196	1087
1.9	1.77021040	1260243	10352	6.9	153.60428	133203	1196
2.0	1.93743376	1387006	11420	7.0	168.50062	146408	1316
2.1	2.11852718	1525210	12587	7.1	184.86104	160931	1448
2.2	2.31487270	1676024	13864	7.2	202.83078	176904	1593
2.3	2.52797845	1840727	15263	7.3	222.56955	194474	1753
2.4	2.75949147	2020720	16795	7.4	244.25306	213799	1929
2.5	3.01121169	2217540	18475	7.5	268.07456	235057	2123
2.6	3.28510732	2432868	20316	7.6	294.24663	258442	2336
2.7	3.58333162	2668549	22336	7.7	323.00312	284167	2571
2.8	3.90824140	2926607	24553	7.8	354.60128	312469	2830
2.9	4.26241725	3209263	26986	7.9	389.32413	343606	3115
3.0	4.64868573	3518954	29658	8.0	427.48303	377863	3428
3.1	5.07014376	3858358	32591	8.1	469.42057	415555	3773
3.2	5.53018536	4230413	35812	8.2	515.51366	457028	4154
3.3	6.03253110	4638346	39352	8.3	566.17703	502663	4572
3.4	6.58126029	5085703	43239	8.4	621.86702	552878	5033
3.5	7.18084652	5576379	47512	8.5	683.08580	608137	5541
3.6	7.83619653	6114655	52206	8.6	750.38595	668947	6100
3.7	8.55269310	6705232	57365	8.7	824.37557	735868	6715
3.8	9.33624199	7353281	63035	8.8	905.72388	809518	7393
3.9	10.19332369	8064481	69269	8.9	995.16736	890575	8140
4.0	11.13105020	8845078	76119	9.0	1093.51660	979788	8962
4.1	12.15722750	9701935	83652	9.1	1201.66372	1077979	9867
4.2	13.28042416	10642600	91932	9.2	1320.59063	1186057	10864
4.3	14.51004681	11675367	101038	9.3	1451.37810	1305020	11963
4.4	15.85642313	12809361	111049	9.4	1595.21577	1435968	13172
4.5	17.33089306	14054610	122058	9.5	1753.41313	1580115	14505
4.6	18.94590910	15422145	134165	9.6	1927.41163	1738794	15972
4.7	20.71514658	16924095	147478	9.7	2118.79808	1913476	17589
4.8	22.65362502	18573799	162122	9.8	2329.31929	2105781	19369
4.9	24.77784144	20385927	178225	9.9	2560.89831	2317492	21330
5.0	27.10591713	22376613	195939	10.0	2815.65225	2550574	23487

* The fourth differences have been modified. They are to be used like ordinary differences for purposes of interpolation. For an explanation of modified differences, see introductory note by L. J. Comrie in B.A.A.S. "Mathematical Tables, Vol. 1."

x	$L_{-1}(x)$	δ^2	$^*\delta^4$	x	$L_{-1}(x)$	δ^2	$^*\delta^4$
0.0	0.63661977	424696	3394	5.1	26.708418	229150	2043
.1	.63874325	426396	3409	5.2	29.281151	251772	2247
.2	.64513070	431511	3452	5.3	32.105655	276645	2472
.3	.65583325	440085	3526	5.4	35.206804	303995	2719
.4	.67093665	452190	3630	5.5	38.611948	334070	2992
0.5	0.69056196	467932	3765	5.6	42.351162	367141	3291
.6	.71486658	487445	3932	5.7	46.457517	403511	3621
.7	.74404565	510897	4133	5.8	50.967383	443508	3984
.8	.77833369	538489	4370	5.9	55.920758	487498	4384
.9	.81800662	570460	4645	6.0	61.361631	535881	4824
1.0	0.86338415	607083	4960	6.1	67.338384	589096	5309
1.1	.91483252	648676	5318	6.2	73.904234	647631	5842
1.2	.97276764	695595	5722	6.3	81.117715	712019	6429
1.3	1.03765871	748248	6176	6.4	89.043214	782847	7075
1.4	1.11003226	807087	6685	6.5	97.751561	860764	7786
1.5	1.19047668	872624	7251	6.6	107.32067	94648	857
1.6	1.27964734	945424	7881	6.7	117.83627	104079	943
1.7	1.37827223	1026120	8580	6.8	129.39265	114454	1038
1.8	1.48715834	1115413	9355	6.9	142.09357	125870	1143
1.9	1.60719856	1214077	10212	7.0	156.05319	138430	1258
2.0	1.73937956	1322973	11160	7.1	171.39711	152251	1384
2.1	1.88479029	1443049	12206	7.2	188.26353	167458	1524
2.2	2.04463151	1575354	13360	7.3	206.80454	184193	1678
2.3	2.22022627	1721042	14632	7.4	227.18748	202609	1847
2.4	2.41303146	1881390	16034	7.5	249.59651	222875	2033
2.5	2.62465054	2057802	17579	7.6	274.23428	245178	2238
2.6	2.85684765	2251825	19280	7.7	301.32384	269724	2464
2.7	3.11156299	2465164	21153	7.8	331.11063	296739	2713
2.8	3.39092998	2699695	23215	7.9	363.86481	326472	2987
2.9	3.69729392	2957485	25484	8.0	399.88371	359199	3289
3.0	4.03323272	3240807	27982	8.1	439.49461	395221	3622
3.1	4.40157959	3552162	30729	8.2	483.05771	434872	3988
3.2	4.80544808	3894304	33753	8.3	530.96954	478518	4391
3.3	5.24825961	4270261	37079	8.4	583.66655	526564	4835
3.4	5.73377375	4683367	40738	8.5	641.62920	579454	5324
3.5	6.26612156	5137286	44764	8.6	705.38638	637679	5863
3.6	6.84984223	5636054	49193	8.7	775.52035	701778	6457
3.7	7.48992344	6184106	54066	8.8	852.07211	772347	7111
3.8	8.19184571	6786326	59426	8.9	937.54733	850040	7831
3.9	8.96163124	7448082	65322	9.0	1030.92296	935579	8624
4.0	9.80589758	8175283	71810	9.1	1133.65438	1029760	9498
4.1	10.73191676	8974429	78947	9.2	1246.68339	1133457	10461
4.2	11.74768022	9852670	86799	9.3	1371.04698	1247635	11522
4.3	12.86197038	10817874	95439	9.4	1507.88692	1373357	12690
4.4	14.08443928	11878695	104944	9.5	1658.46043	1511795	13977
4.5	15.42569514	13044658	115401	9.6	1824.15188	1664237	15396
4.6	16.89739758	14326240	126910	9.7	2006.48571	1832104	16958
4.7	18.51236241	15734970	139569	9.8	2207.14057	2016962	18679
4.8	20.28467695	17283533	153502	9.9	2427.96505	2220537	20577
4.9	22.22982681	18985888	168833	10.0	2670.99490	2444728	22671
5.0	24.36483555	20857395	185703				

x	$L_{-2}(x)$	$xL_{-2}(x)$	$\delta^2(xL_{-2})$	$*\delta^4$ (xL_{-2})
0.0	$-\infty$	−.63661977	425262	10180
.1	−6.34493460	−.63449346	430368	10253
.2	−3.14031737	−.62806347	445756	10472
.3	−2.05725309	−.61717593	471647	10841
.4	−1.50392978	−.60157191	508411	11363
0.5	−1.16176758	−.58088379	556571	12046
.6	−.92438326	−.55462995	616812	12897
.7	−.74601144	−.52220801	689986	13928
.8	−.60360774	−.48288620	777129	15151
.9	−.48421455	−.43579310	879465	16581
1.0	−.37990535	−.37990535	998429	18236
1.1	−.28548483	−.31403331	1135680	20137
1.2	−.19733707	−.23680448	1293123	22306
1.3	−.11280340	−.14664442	1472933	24770
1.4	−.02982502	−.04175502	1677581	27560
1.5	+.05327345	+.07991018	1909864	30710
1.6	.13792126	.22067402	2172940	34258
1.7	.22539250	.38316726	2470365	38247
1.8	.31686897	.57036414	2806138	42725
1.9	.41348548	.78562241	3184749	47747
2.0	.51636408	1.03272817	3611234	53374

x	$L_{-2}(x)$	δ^2	$*\delta^4$
2.1	.62664108	857133	5649
2.2	.74548940	980039	7406
2.3	.87413811	1110333	9075
2.4	1.01389016	1249696	10717
2.5	1.16613917	1399779	12375
2.6	1.33238596	1562248	14090
2.7	1.51425524	1738825	15890
2.8	1.71351276	1931315	17806
2.9	1.93208343	2141640	19862
3.0	2.17207051	2371862	22085
3.1	2.43577621	2624209	24500
3.2	2.72572399	2901400	27132
3.3	3.04468278	3205174	30009
3.4	3.39569331	3539314	33161
3.5	3.78209699	3906679	36618
3.6	4.20756745	4310731	40414
3.7	4.67614523	4755275	44585
3.8	5.19227576	5244490	49171
3.9	5.76085120	5782972	54218
4.0	6.38725635	6375776	59770
4.1	7.07741927	7028466	65883
4.2	7.83786685	7747168	72612
4.3	8.67578611	8538622	80023
4.4	9.59909158	9410255	88185
4.5	10.61649961	10370245	97174
4.6	11.73761009	11427599	107077
4.7	12.97299656	12592237	117984
4.8	14.33430540	13875090	130002
4.9	15.83436514	15288198	143240
5.0	17.48730686	16844827	157828

x	$L_{-2}(x)$	δ^2	$*\delta^4$
5.1	19.308697	185596	1739
5.2	21.315683	204486	1916
5.3	23.527155	225296	2111
5.4	25.963922	248221	2326
5.5	28.648910	273477	2563
5.6	31.607376	301300	2824
5.7	34.867141	331954	3112
5.8	38.458860	365725	3429
5.9	42.416305	402932	3778
6.0	46.776681	443924	4163
6.1	51.580981	489087	4587
6.2	56.874367	538845	5054
6.3	62.706599	593669	5570
6.4	69.132499	654072	6137
6.5	76.212472	720625	6763
6.6	84.013069	793954	7452
6.7	92.607621	874750	8212
6.8	102.07692	96377	905
6.9	112.51000	106187	997
7.0	124.00494	116995	1099
7.1	136.66983	128904	1211
7.2	150.62377	142027	1335
7.3	165.99798	156488	1471
7.4	182.93707	172422	1621
7.5	201.60038	189980	1786
7.6	222.16348	209328	1969
7.7	244.81987	230649	2170
7.8	269.78274	254143	2391
7.9	297.28704	280034	2636
8.0	327.59168	308565	2905
8.1	360.98198	340007	3201
8.2	397.77234	374656	3528
8.3	438.30926	412840	3889
8.4	482.97458	454921	4286
8.5	532.18912	501297	4724
8.6	586.41663	552406	5207
8.7	646.16821	608733	5739
8.8	712.00711	670810	6326
8.9	784.55410	739225	6973
9.0	864.49335	814627	7686
9.1	952.57887	897730	8472
9.2	1049.64169	989322	9338
9.3	1156.59774	1090271	10294
9.4	1274.45649	1201533	11347
9.5	1404.33058	1324165	12508
9.6	1547.44631	1459328	13788
9.7	1705.15532	1608307	15199
9.8	1878.94740	1772514	16754
9.9	2070.46462	1953509	18468
10.0	2281.51693	2153006	20354

x	$H_0(x)$	$*\delta^2$	x	$H_0(x)$	$*\delta^2$
0.0	0.0000000	0000000	5.0	−0.1852168	34710
0.1	.0635913	−4242	5.1	.2005919	35245
0.2	.1267590	8450	5.2	.2124483	35459
0.3	.1890829	12590	5.3	.2207647	35355
0.4	.2501497	16630	5.4	.2255513	34937
0.5	0.3095559	−20537	5.5	−0.2268499	34212
0.6	.3669114	24281	5.6	.2247327	33193
0.7	.4218424	27830	5.7	.2193014	31889
0.8	.4739944	31159	5.8	.2106861	30319
0.9	.5230350	34240	5.9	.1990435	28498
1.0	0.5686566	−37049	6.0	−0.1845553	26446
1.1	.6105787	39566	6.1	.1674264	24185
1.2	.6485500	41770	6.2	.1478824	21738
1.3	.6823503	43645	6.3	.1261676	19130
1.4	.7117925	45177	6.4	.1025422	16386
1.5	0.7367235	−46356	6.5	−0.0772802	13533
1.6	.7570255	47174	6.6	.0506665	10598
1.7	.7726168	47626	6.7	−0.0229940	7609
1.8	.7834523	47710	6.8	+0.0054389	4595
1.9	.7895236	47427	6.9	.0343315	+1584
2.0	0.7908588	−46783	7.0	0.0633830	−1397
2.1	.7875222	45785	7.1	.0922958	4320
2.2	.7796135	44442	7.2†	.1207783	7160
2.3	.7672665	42769	7.3	.1485467	9889
2.4	.7506485	40781	7.4	.1753286	12487
2.5	0.7299577	−38497	7.5	0.2008648	−14927
2.6	.7054223	35938	7.6†	.2249116	17191
2.7	.6772977	33127	7.7	.2472429	19256
2.8	.6458646	30089	7.8	.2676524	21113
2.9	.6114264	26851	7.9	.2859549	22737
3.0	0.5743061	−23441	8.0	0.3019881	−24122
3.1	.5348444	19890	8.1	.3156137	25258
3.2	.4933957	16228	8.2	.3267183	26130
3.3	.4503257	12486	8.3	.3352147	26744
3.4	.4060080	8697	8.4†	.3410417	27083
3.5	0.3608208	−4893	8.5	0.3441653	−27162
3.6	.3151440	−1105	8.6	.3445775	26974
3.7	.2693559	+2635	8.7	.3422972	26522
3.8	.2238298	6295	8.8	.3373694	25818
3.9	.1789312	9846	8.9†	.3298641	24872
4.0	0.1350146	13258	9.0	0.3198760	−23693
4.1	.0924208	16505	9.1†	.3075225	22293
4.2	.0514740	19560	9.2	.2929435	20694
4.3	+0.0124793	22400	9.3	.2762985	18907
4.4	−0.0242798	25003	9.4	.2577659	16954
4.5	−0.0585433	27348	9.5	0.2375406	−14855
4.6	.0900771	29419	9.6†	.2158321	12629
4.7	.1186742	31202	9.7†	.1928626	10307
4.8	.1441567	32683	9.8†	.1688638	7904
4.9	.1663766	33855	9.9†	.1440756	5444
5.0	−0.1852168	34710	10.0	0.1187437	−2956

† The roundings of these entries disagree with those of G. N. Watson.

* The second differences have been modified. In practice, they are to be used exactly like ordinary differences for purposes of interpolation.

x	$H_{-1}(x)$	$*\delta^2$	x	$H_{-1}(x)$	$*\delta^2$
0.0	0.6366198	−42475	5.0	−0.1711922	6955
0.1	.6344991	42306	5.1	.1362200	3746
0.2	.6281541	41798	5.2	.1008732	+543
0.3	.6176355	40956	5.3	.0654714	−2623
0.4	.6030272	39788	5.4	−0.0303308	5727
0.5	0.5844460	−38303	5.5	+0.0042387	−8741
0.6	.5620401	36514	5.6	.0379363	11639
0.7	.5359881	34437	5.7	.0704726	14396
0.8	.5064973	32089	5.8	.1015722	16989
0.9	.4738023	29490	5.9	.1309763	19397
1.0	0.4381624	−26663	6.0	0.1584445	−21599
1.1	.3998601	23631	6.1	.1837569	23578
1.2	.3591980	20420	6.2	.2067158	25319
1.3	.3164966	17058	6.3	.2271474	26807
1.4	.2720918	13573	6.4	.2449032	28032
1.5	0.2263313	−9995	6.5	0.2598607	−28986
1.6	.1795725	6353	6.6	.2719246	29663
1.7	.1321790	−2679	6.7	.2810275	30058
1.8	.0845176	+996	6.8	.2871296	30173
1.9	.0369553	4642	6.9	.2902197	30007
2.0	−0.0101440	8228	7.0	0.2903140	−29567
2.1	.0564220	11725	7.1	.2874567	28857
2.2	.1015298	15103	7.2†	.2817184	27888
2.3	.1451301	18334	7.3	.2731958	26672
2.4	.1869000	21392	7.4	.2620103	25222
2.5	−0.2265344	24253	7.5	0.2483067	−23554
2.6	.2637476	26893	7.6	.2322513	21683
2.7	.2982759	29290	7.7	.2140310	19635
2.8	.3298800	31427	7.8	.1938502	17423
2.9	.3583465	33286	7.9	.1719296	15074
3.0	−0.3834898	34854	8.0	0.1485037	−12610
3.1	.4051532	36119	8.1	.1238185	10054
3.2	.4232106	37072	8.2	.0981291	7437
3.3	.4375665	37707	8.3†	.0716968	4770
3.4	.4481577	38022	8.4	.0447877	−2095
3.5	−0.4549525	38015	8.5†	+0.0176688	+571
3.6	.4579518	37689	8.6†	−0.0093935	3206
3.7	.4571879	37050	8.7	.0361364	5779
3.8	.4527247	36104	8.8	.0623028	8270
3.9	.4446564	34864	8.9	.0876443	10654
4.0	−0.4331069	33341	9.0	−0.1119226	12913
4.1	.4182282	31552	9.1	.1349124	15024
4.2	.4001989	29513	9.2†	.1564028	16969
4.3	.3792224	27245	9.3†	.1761996	18729
4.4	.3555253	24770	9.4	.1941271	20299
4.5	−0.3293546	22110	9.5	−0.2100287	21646
4.6	.3009758	19291	9.6	.2237696	22783
4.7	.2706703	16338	9.7	.2352365	23682
4.8	.2387330	13279	9.8	.2443396	24347
4.9	.2054693	10142	9.9	.2510124	24770
5.0	−0.1711922	6955	10.0	−0.2552127	24948

x	$H_{-2}(x)$	$xH_{-2}(x)$	$*\delta^2$
0.0	$-\infty$	−0.6366198	−42543
0.1	−6.3873760	.6387376	42034
0.2	3.2252013	.6450403	40513
0.3	2.1845869	.6553761	38002
0.4	1.6737364	.6694946	34537
0.5	−1.3741005	−0.6870502	−30168
0.6	1.1793455	.7076073	24958
0.7	1.0437801	.7306460	18980
0.8	0.9444629	.7555704	12323
0.9	0.8585736	.7817163	−5083
1.0	−0.8083617	−0.8083617	+2635
1.1	.7588518	.8347370	10717
1.2	.7166968	.8600361	19042
1.3	.6795607	.8834289	27486
1.4	.6457666	.9040732	35921
1.5	−0.6140853	−0.9211280	44217
1.6	.5836038	.9337660	52245
1.7	.5536394	.9411869	59876
1.8	.5236832	.9426297	66986
1.9	.4933609	.9373857	73456
2.0	.4624050	.9248100	79172

x	$H_{-2}(x)$	$*\delta^2$
2.0	−0.4624050	8276
2.1	.4306347	9345
2.2	.3979410	9814
2.3	.3642753	9775
2.4	.3296402	9298
2.5	−0.2940823	8445
2.6	.2576858	7269
2.7	.2205675	5817
2.8	.1828719	4130
2.9	.1447668	2249
3.0	−0.1064397	+209
3.1	.0680940	−1955
3.2	−0.0299455	4208
3.3	+0.0077812	6520
3.4	.0448553	8860
3.5	0.0810435	−11197
3.6	.1161125	13504
3.7	.1498321	15754
3.8	.1819779	17920
3.9	.2123336	19979
4.0	0.2406938	−21907
4.1	.2668661	23684
4.2	.2906730	25291
4.3	.3119544	26709
4.4	.3305685	27924
4.5	0.3463942	−28923
4.6	.3593317	29695
4.7	.3693041	30230
4.8	.3762580	30523
4.9	.3801640	30570
5.0	0.3810176	−30369

x	$H_{-2}(x)$	$*\delta^2$
5.0	0.3810176	−30369
5.1	.3788389	29921
5.2	.3736726	29229
5.3	.3655878	28298
5.4	.3546775	27137
5.5	0.3410576	−25754
5.6	.3248661	24163
5.7	.3062618	22377
5.8	.2854232	20411
5.9	.2625464	18284
6.0	0.2378438	−16013
6.1	.2115422	13619
6.2	.1838805	11123
6.3	.1551080	8548
6.4	.1254818	5915
6.5	0.0952646	−3250
6.6	.0647227	−574
6.7	.0341231	+2089
6.8	+0.0037317	4714
6.9	−0.0261894	7278
7.0	−0.0553841	9760
7.1	.0836048	12138
7.2	.1106140	14391
7.3	.1361867	16498
7.4	.1601125	18444
7.5	−0.1821973	20210
7.6	.2022646	21783
7.7	.2201574	23146
7.8	.2357396	24295
7.9	.2488966	25213
8.0	−0.2595366	25898
8.1	.2675911	26345
8.2	.2730157	26547
8.3	.2757899	26511
8.4	.2759174	26230
8.5	−0.2734262	25718
8.6	.2683674	24974
8.7	.2608153	24006
8.8	.2508665	22828
8.9	.2386385	21451
9.0	−0.2242687	19889
9.1	.2079132	18154
9.2	.1897451	16271
9.3	.1699524	14250
9.4	.1487367	12115
9.5	−0.1263114	9885
9.6	.1028989	7581
9.7	.0787293	5229
9.8	.0540373	2845
9.9	.0290610	+453
10.0	−0.0040391	−1923

THE WIENER RMS (ROOT MEAN SQUARE) ERROR CRITERION IN FILTER DESIGN AND PREDICTION

By Norman Levinson

In the process of gathering or transmitting information by mechanical or electrical means the signal that contains the information frequently becomes distorted. Among the diverse sources of distortion there may be tracking errors, crosstalk, thermal noise, and poor characteristics of pickup, transmitting, or receiving equipment. When the distortion has random statistical features it is called noise.

The modification of a signal is sometimes necessary in order to remove the noise and recapture the original message. This process is called filtering. The determination of how much of the noise can be separated from the message contained in a signal is by no means simple. Cases exist where a crude filter will perform as well as an extremely complicated one. There are cases where the very best results require an elaborate filter but where only slightly inferior results can be obtained by using comparatively simple filters

In this article a method will be presented for determining quantitatively the extent to which message and noise can be separated. Also will be given a method of designing a filter to carry out this separation. The close of the article will consider the problem of filtering and predicting simultaneously. The root mean square error approach used here is an approximation to and a simplification of the transcendental case developed by N. Wiener.

Wiener's work appeared in a book of limited circulation in February 1942. An independent and similar but by no means identical work by Kolmogoroff had already appeared in Bull. Acad. Sciences USSR pp 3–14, 1941.

A few months after Wiener's work appeared, the author, in order to facilitate computational procedure, worked out an approximate, and one might say, mathematically trivial procedure. This procedure is essentially that which appears in §2, 3 and 6 of the present paper. It is, actually classical least squares

The basic idea underlying Wiener's work, rather than the intricate mathematical procedure for solving the transcendental problem, influenced work on smoothing and prediction in fire control work. Various classified documents have appeared some of which have lately been declassified. Of these the author has seen and been influenced (so far as minimizing the expression (35) of this paper is concerned) by the work of Phillips and Weiss, Theoretical Calculation on Best Smoothing of Position Data for Gunnery Prediction, N D R C Report 532, February 1944.

Another relevant report which the author knows only by title is by Blackman, Bode and Shannon, Monograph on Data Smoothing and Prediction in Fire Control Systems, N D R C Report, Feb. 1946.

A short heuristic account by the author of the transcendental work of Wiener will be published shortly. The detailed and mathematically rigorous account by Wiener has not as yet been published in a publicly available form.

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1. **Linear Filters.** Here the discussion will be limited to linear filtering devices. The behavior of a linear filter may be expressed in terms of its impedance function. This impedance function gives the characteristic of the filter in terms of the relationship of the amplitude and phase of the output to those of any sinusoidal input. A dual way of indicating the behavior of the filter is in terms of the output corresponding to an input which is a unit-step function. If, when the input is $\begin{cases} 1, t > 0 \\ 0, t < 0 \end{cases}$, we denote the output by $A(t)$, then corresponding to any input $f(t)$ the output

$$F(t) = \int_0^\infty A'(\tau) f(t - \tau)\, d\tau + A(0) f(t). \tag{1}$$

By $A(0)$ is meant the limit of $A(t)$ as t approaches zero through positive values.

It is useful for many purposes to approximate to the integral in Eq. (1) by a sum. We have from Eq. (1) approximately, if h is small,

$$F(t) = h \sum_{n=1}^{\infty} A'(nh) f(t - nh) + A(0) f(t). \tag{2}$$

In case $A'(\tau)$ is small when τ is large the tail of the infinite series can be discarded. Using the notation $A_n = hA'(nh)$, $n > 0$, and $A_0 = A(0)$ we have approximately, for some suitably chosen M

$$F(t) = \sum_{n=0}^{M} A_n f(t - nh).$$

This last result states that the output is given approximately by a certain linear combination of the input and a number of its past values. Or, to put it differently, $F(t)$ is given approximately by a weighted sum of a number of past values of the input. In case $F(t)$ and $f(t)$ are adequately determined by their values at $t = kh$, we find

$$F(kh) = \sum_{n=0}^{M} A_n f[(k - n)h].$$

Calling $F(kh)$, F_k and $f[(k - n)h]$, f_{k-n}, we have

$$F_k = \sum_{n=0}^{M} A_n f_{k-n}. \tag{3}$$

2. **Minimization of RMS Error.** Let $f(t)$ be a signal containing a message $g(t)$ and noise. Clearly the noise is given by $f(t) - g(t)$.

Let us consider the output of an electrical circuit with input $f(t)$. If the circuit has the response $A(t)$ to a unit-step function, then referring to Eq. (1) we see that the output, with input $f(t)$, is given by

$$F(t) = \int_0^\infty A'(\tau) f(t - \tau)\, d\tau + A(0) f(t).$$

Our goal is to have $F(t)$ approximate as closely as possible the message $g(t)$. That is, we want to minimize $[F(t) - g(t)]$. As a criterion for measuring the difference between $F(t)$ and $g(t)$ we shall take

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} [F(t) - g(t)]^2 \, dt.$$

This is clearly the square of the *rms* value of $F(t) - g(t)$. The exact procedure for determining $A'(\tau)$ from the requirement that the *rms* value of $F(t) - g(t)$ be a minimum leads to an integral equation which can be solved by rather involved analysis.[1]

To avoid the transcendental analysis arising when the signal is treated as a continuous function, $f(t)$, we choose a time interval h sufficiently small so that $f(t)$ is well characterized by its values at the points $t = kh$, where k assumes integral values. If we denote $f(kh)$ by b_k then we can regard a signal as a sequence b_k. The message contained in the signal we shall denote by the sequence a_k, and noise, by the sequence of differences, $b_k - a_k$. It is our purpose to find the best way to treat the signal, that is, the b_k, so as to obtain the information, the a_k.

Let us try to determine the nature of a linear filter which, with input b_k, will have an output as close as possible to a_k. Using Eq. (3) we see that our problem is to determine the numbers A_n so that the

$$\epsilon_k = a_k - \sum_{n=0}^{M} A_n b_{k-n} \tag{4}$$

are as small as possible. What we shall do to try to make the ϵ_k as small as possible, is to require that the A_n be so chosen that the average of the sum of the square of ϵ_k should be a minimum. This is equivalent to requiring that the *rms* of the ϵ_k be a minimum.

Stated in formula we want to choose A_n so that,

$$I = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} \left(a_k - \sum_{n=0}^{M} A_n b_{k-n} \right)^2 \tag{5}$$

should be a minimum. Equation (5) assumes a much simpler form if we introduce autocorrelation functions. The autocorrelation function is defined as

$$R_a(k) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{l=-N}^{N} a_l a_{l-k}$$

It is an even function, that is, it has the property

$$R_a(k) = R_a(-k).$$

[1] An account of a very similar theory for the case of prediction rather than filtering is given by N. Levinson, "An Expository Account of Wiener's Theory of Prediction," AMP Note No. 20, June, 1945. A completely rigorous account will appear in a coming publication of Wiener probably as a volume in the series, Mathematical Surveys.

We shall also be concerned with the autocorrelation function

$$R_b(k) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{l=-N}^{N} b_l b_{l-k},$$

and the cross-correlation function

$$R_{ba}(k) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{l=-N}^{N} a_l b_{l-k}.$$

The cross-correlation function is not necessarily an even function of k.

It frequently happens that the message and noise are completely uncorrelated. In this case $R_{b-a,a}(k) = 0$ for all (k). Since

$$R_{ba}(k) = R_{b-a,a}(k) + R_a(k)$$

we see that if the message and noise have zero correlation $R_{ba}(k) = R_a(k)$.

When $R_{ba}(k)$ is itself zero, the signal and message have no correlation. This means that the noise cancels the message completely and leaves only a random residue, making it impossible to separate any part of the message from the signal by a linear device. Thus $R_{ba}(k) = 0$ is the worst situation that can arise.

We can write Eq. (5) as

$$I = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} a_k^2 - 2 \sum_{n=0}^{M} A_n \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} a_k b_{k-n} + \sum_{n=0}^{M} \sum_{m=0}^{M} A_n A_m \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} b_{k-n} b_{k-m}.$$

Using the auto- and cross-correlation functions we have

$$I = R_a(0) - 2 \sum_{n=0}^{M} A_n R_{ba}(n) + \sum_{n,m=0}^{M} A_n A_m R_b(m-n). \tag{6}$$

If the A_n are chosen so as to make I a minimum we must have

$$\frac{\partial I}{\partial A_k} = 0, \qquad k = 0, 1, \cdots, M.$$

Thus

$$\frac{\partial I}{\partial A_k} = -2R_{ba}(k) + 2 \sum_{n=0}^{M} A_n R_b(k-n) = 0.$$

Or

$$\sum_{n=0}^{M} A_n R_b(k-n) = R_{ba}(k), \qquad k = 0, 1, \cdots; M. \tag{7}$$

We have derived Eqs. (7) as a necessary condition that the A_n make I a minimum. We shall have occasion to prove later, that with the A_n obtained from Eqs. (7), I actually assumes its minimum value.

Equations (7) are a linear system of M + 1 equations in the M + 1 unknowns

A_n. From Eqs. (7) we see that the determination of A_n is dependent on the autocorrelation function of the b's and on the cross correlation of the a's and b's. It does not depend on the a's and b's directly as such. Thus, while the a's and b's may differ from one run to another, if the correlation functions do not, then a set of A_n's can be chosen once and for all which will work for all the runs. In other words, it is necessary for the sequences a_k and b_k to be elements of a stationary random process.

The advantage of dealing with the discrete sequences a_k and b_k rather than with the continuous functions $g(t)$ and $f(t)$ is that in the discrete case we face simply the linear algebraic problem [Eqs. (7)] as contrasted to an integral equation in the continuous case.

Using Eqs. (7) in (6) we see that the minimum value of I, I_m, is given by

$$I_m = R_a(0) - \sum_{n=0}^{M} A_n R_{ba}(n). \tag{8}$$

The sum, $\Sigma A_n R_{ba}(n)$, on the right-hand side of Eqs. (7), cannot be negative since the choice $A_n = 0$ would in that case reduce I_m. This is an impossibility since I_m is already a minimum. Thus $\Sigma A_n R_{ba}(n) \geqq 0$. The worst case that can arise is for this expression to be zero. This happens when all the $R_{ba}(n)$ are zero. For the $R_{ba}(n)$ to be zero means that the signal and the message are completely incoherent. In this worst of all cases, $I_m = R_a(0)$ This suggests normalizing Eqs. (8) by dividing by $R_a(0)$:

$$\frac{I_m}{R_a(0)} = 1 - \sum_{n=0}^{M} A_n \frac{R_{ba}(n)}{R_a(0)}.$$

If we now call $I_m/[R_a(0)]$, V, and if we set

$$\frac{R_{ba}(n)}{R_a(0)} = \gamma_n,$$

then we have

$$V = 1 - E_M, \tag{9}$$

where

$$E_M = \sum_{n=0}^{M} A_n \gamma_n. \tag{10}$$

Obviously $V \leqq 1$ since $I_m \leqq R_a(0)$. On the other hand, since I_m is the average of a sum of squares, $I_m \geqq 0$, and thus $V \geqq 0$. We see then that

$$0 \leqq E_M \leqq 1.$$

The closer E_M is to one, the smaller is the *rsm* value of ϵ_k, that is, the better the separation of the message from the noise. The value of E_M increases with M. Ordinarily, increasing M beyond a certain point will increase E_M only very

slightly. There will be a value $\bar{E} = \lim_{M\to\infty} E_M$, beyond which it is impossible to increase E_M. If $\bar{E}$ is small compared to one, it means that even the best linear filter can effectuate only a poor separation of the noise from the message; if $\bar{E}$ is close to one a considerable separation can be attained. The *rms* of ϵ_k for a given filter can be compared with $\sqrt{R_s(0)}\sqrt{1-\bar{E}}$. If this *rms* value is almost as small as $\sqrt{R_s(0)}\sqrt{1-\bar{E}}$, then the filter is close to optimum.

In finding E_M we use the A_n as found from Eq. (7). It is convenient now to set

$$\frac{R_b(n)}{R_s(0)} = r_n .$$

We can write Eqs. (7) as

$$\sum_{n=0}^{M} A_n r_{k-n} = \gamma_k , \qquad k = 0, 1, \cdots, M. \tag{11}$$

In practice it is usually impossible to build a filter with the A_n exactly as required. Moreover, over-all considerations may make it undesirable to do so. In case the filter selected has its characteristic response given by the sequence B_n rather than by the desired A_n the value of I will be affected. Let the difference $B_n - A_n$ be denoted by δ_n. In this case we have

$$\begin{aligned} I &= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} \left(a_k - \sum_{n=0}^{M} B_n b_{k-n} \right)^2 \\ &= R_a(0) - 2\sum_{n=0}^{M} B_n R_{ba}(n) + \sum_{n,m=0}^{M} B_n B_m R_b(n-m). \end{aligned}$$

Using $B_n = A_n + \delta_n$ and dividing by $R_a(0)$ we find, denoting $I/[R_a(0)]$ by V, that

$$\begin{aligned} V = 1 &- 2\sum_{n=0}^{M} A_n \gamma_n + \sum_{n,m=0}^{M} A_n A_m r_{n-m} - 2\sum_{n=0}^{M} \delta_n \gamma_n \\ &+ 2\sum_{n,m=0}^{M} A_n \delta_m r_{n-m} + \sum_{n,m=0}^{M} \delta_n \delta_m r_{n-m} . \end{aligned}$$

Using Eqs. (11) we find

$$V = 1 - \sum_{n=0}^{M} A_n \gamma_n + \sum_{n,m=0}^{M} \delta_n \delta_m r_{n-m} .$$

Using Eqs. (10) this becomes

$$V = 1 - E_M + \sum_{n,m=0}^{M} \delta_n \delta_m r_{n-m} . \tag{12}$$

Thus the effect of using B_n instead of A_n is to increase V by

$$J = \sum_{n,m=0}^{M} \delta_n \delta_m r_{n-m} . \tag{13}$$

It is not difficult to show that $J \geqq 0$. In fact

$$r_{n-m} = \frac{1}{R_a(0)} \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} b_{k+n} b_{k+m}.$$

Thus, heuristically we have from Eqs. (13)

$$J = \frac{1}{R_a(0)} \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} \sum_{n,m=0}^{M} \delta_n \delta_m b_{k+n} b_{k+m}$$

$$= \frac{1}{R_a(0)} \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} \left(\sum_{n=0}^{M} b_{k+n} \delta_n \right)^2.$$

Since the sum of squared terms is positive, we have $J \geqq 0$.

We have from Eqs. (12)

$$I = R_a(0)\,(1 - E_M + J).$$

Since $J \geqq 0$ we see that the minimum value of I, I_m is assumed when $J = 0$. When $\delta_n = 0$ we have $J = 0$. Thus choosing the A_n as the solution of Eqs. (11) does cause I to assume its minimum value.

3. Determination of the Weighting Function. We now turn our attention to a method for solving Eqs. (11) for the A_n. We recall that Eqs. (11) state

$$\sum_{n=0}^{M} A_n r_{k-n} = \gamma_k, \qquad k = 0, 1, \cdots, M. \tag{11}$$

From the definition of $r_k = [R_b(k)]/[R_a(0)]$ we see that since the $R_b(k)$ sequence is even, so is the r_k. This simplifies the process of solving Eqs. (11) for the A_n as we shall now see. In case $M = 3$, for example, we have

$$\begin{aligned} A_0 r_0 + A_1 r_1 + A_2 r_2 + A_3 r_3 &= \gamma_0 \\ A_0 r_1 + A_1 r_0 + A_2 r_1 + A_3 r_2 &= \gamma_1 \\ A_0 r_2 + A_1 r_1 + A_2 r_0 + A_3 r_1 &= \gamma_2 \\ A_0 r_3 + A_1 r_2 + A_2 r_1 + A_3 r_0 &= \gamma_3. \end{aligned}$$

Adding the first equation to the last, the second equation to the next to last, and in the general case proceeding further in this way, we have

$$\begin{aligned} (A_0 + A_3)(r_0 + r_3) + (A_1 + A_2)(r_1 + r_2) &= \gamma_0 + \gamma_3 \\ (A_0 + A_3)(r_1 + r_2) + (A_1 + A_2)(r_0 + r_1) &= \gamma_1 + \gamma_2. \end{aligned}$$

This is a pair of equations for the two unknowns $A_0 + A_3$ and $A_1 + A_2$.

Again subtracting the last equation from the first and the next to the last from the second, we get

$$\begin{aligned} (A_0 - A_3)(r_0 - r_3) + (A_1 - A_2)(r_1 - r_2) &= \gamma_0 - \gamma_3 \\ (A_0 - A_3)(r_1 - r_2) + (A_1 - A_2)(r_0 - r_1) &= \gamma_1 - \gamma_2. \end{aligned}$$

Here again we have two equations for the two unknowns $A_0 - A_3$ and $A_1 - A_2$.

Solving each of these two systems of equations, we get $A_0 + A_3$, $A_1 + A_2$, $A_0 - A_3$, and $A_1 - A_2$. Adding the first and the third of these quantities and

dividing by two we get A_0, subtracting we get A_3. Proceeding similarly with the second and fourth we find A_1 and A_2.

The amount of work in finding the A_n is considerably reduced by this device. Thus when $M = 7$ there are eight unknowns and solving Eqs. (11) is a formidable computation. By the procedure just discussed, this case is reduced to solving two systems of equations each in four unknowns. The latter problem is very tractable.

In case M is even, the device still works in a slightly modified form. Here we add the middle equation to itself and subtract it from itself. We then obtain two systems of equations, one with $[(M/2) + 1]$ and the other with $(M/2)$ unknowns.

We now turn to a procedure for getting the A_n by an iteration process. The method which we are about to develop will give some insight on the number of weights A_n needed for a good filter.

We have found that a measure of the effectiveness of the filter output

$$\sum_{k=0}^{M} A_k b_{n-k} \tag{14}$$

in representing the message a_n, was given by

$$E_M = \sum_{k=0}^{M} A_k \gamma_k .$$

The closer E_M is to one the more effectively (14) represents a_n.

It is an important practical question to decide how large to make M. Unless E_M increases appreciably when M is increased, it is not worth while to increase M. In practice this makes desirable a procedure which gives us E_1, E_2, E_3, etc., without undue computational difficulty. To distinguish between the various values A_n assumes as M changes, we introduce the more specific notation, $A_n^{(M)}$.

Thus Eqs. (11) and (10) become

$$\sum_{n=0}^{M} A_n^{(M)} r_{k-n} = \gamma_k , \qquad (k = 0, 1, \cdots, M), \tag{15}$$

and

$$E_M = \sum_{k=0}^{M} A_k^{(M)} \gamma_k . \tag{16}$$

We shall now set up an iterative process by means of which we can proceed easily from $A_n^{(M)}$ to $A_n^{(M+1)}$. We introduce first an auxiliary sequence $C_k^{(M)}$ which we specify as follows:

$$\left.\begin{aligned} C_0^{(0)} &= r_1/r_0 \\ C_0^{(M)} \left(r_0 - \sum_{k=0}^{M-1} C_k^{(M-1)} r_{M-k} \right) &= r_{M+1} - \sum_{k=1}^{M} C_{k-1}^{(M-1)} r_k \end{aligned}\right\} \tag{17}$$

$$C_k^{(M)} = C_{k-1}^{(M-1)} - C_0^{(M)} C_{M-k}^{(M-1)}, \qquad k = 1, 2, \cdots, M. \tag{18}$$

Thus knowing $C_k^{(M-1)}$, $k = 0, 1, \cdots, M-1$, we are able from Eq. (17) to find $C_0^{(M)}$ and then from Eq. (18) to find $C_k^{(M)}$, $k = 1, \cdots, M$.

Having determined the $C_k^{(M)}$, we find $A_{M+1}^{(M+1)}$ from the $C_k^{(M)}$ and $A_k^{(M)}$ by use of

$$\left.\begin{aligned} A_0^{(0)} &= \gamma_0/r_0 \\ A_{M+1}^{(M+1)}\left(r_0 - \sum_{k=0}^{M} C_k^{(M)} r_{M+1-k}\right) &= \gamma_{M+1} - \sum_{k=0}^{M} A_k^{(M)} r_{M+1-k}. \end{aligned}\right\} \tag{19}$$

Also

$$A_k^{(M+1)} = A_k^{(M)} - C_k^{(M)} A_{M+1}^{(M+1)}, \qquad k = 0, 1, \cdots, M. \tag{20}$$

We also have

$$\left.\begin{aligned} E_0 &= \gamma_0^2/r_0 \\ E_{M+1} &= E_M + A_{M+1}^{(M+1)}\left(\gamma_{M+1} - \sum_{k=0}^{M} C_k^{(M)} \gamma_k\right). \end{aligned}\right\} \tag{21}$$

Equations (21) are an immediate consequence of using Eq. (20) in the formula (16) for E_{M+1}. Thus on ascertaining $C_k^{(M)}$ and $A_{M+1}^{(M+1)}$ we can find at once how much larger E_{M+1} will be compared with E_M. This can be done even before using Eq. (20) to compute $A_k^{(M+1)}$, $k \leqq M$.

We now proceed with the proofs of Eqs. (19) and (20). We must show that with the $A_k^{(M)}$ determined from these equations, Eq. (15) is satisfied. First let us see what Eq. (20) gives when used in Eq. (15) with M replaced by $M+1$. We have

$$\begin{aligned} &\sum_{n=0}^{M} (A_n^{(M)} - C_n^{(M)} A_{M+1}^{(M+1)})\, r_{k-n} + A_{M+1}^{(M+1)} r_{k-M-1} \\ &\qquad = \gamma_k, \qquad k = 0, 1, \cdots, M+1 \end{aligned} \tag{22}$$

Using Eq. (15) we obtain from the above with $k < M+1$,

$$\sum_{n=0}^{M} C_n^{(M)} r_{k-n} = r_{M+1-k}, \qquad g = 0, \qquad 1, \cdots, M. \tag{23}$$

For $k = M+1$ we get Eq. (19) from Eq. (22). Thus eqs. (19) and (20) hinge on Eq. (23), which we proceed to prove by induction.

We now use Eq. (18) in Eq. (23) and obtain

$$\sum_{n=1}^{M} (C_{n-1}^{(M-1)} - C_0^{(M)} C_{M-n}^{(M-1)}) r_{k-n} + C_0^{(M)} r_k = r_{M+1-k}, \qquad k = 0, 1, \cdots, M.$$

This can be written as

$$\begin{aligned} \sum_{n=0}^{M-1} C_n^{(M-1)} r_{(k-1)-n} &= r_{M-(k-1)} + C_0^{(M)} \left(\sum_{n=1}^{M} C_{M-n}^{(M-1)} r_{k-n} - r_k\right), \\ k &= 1, 2, \cdots, M, \end{aligned} \tag{24}$$

and Eq. (17). But Eq. (23) with M replaced by $M - 1$ reduces Eq. (24) to

$$\sum_{n=1}^{M} C_{M-n}^{(M-1)} r_{k-n} = r_k, \qquad k = 1, 2, \cdots, M. \tag{25}$$

Replacing n by $M - m$, Eq. (25) becomes

$$\sum_{m=0}^{M-1} C_m^{(M-1)} r_{M-k-m} = r_k, \qquad k = 1, 2, \cdots, M. \tag{26}$$

If we set $M - k = j$ in Eq. (26) we get

$$\sum_{m=0}^{M-1} C_m^{(M-1)} r_{j-m} = r_{M-j}, \qquad j = 0, 1, \cdots, M - 1. \tag{27}$$

But Eq. (27) is the same as Eq. (23) with the index M replaced by $M - 1$. Thus Eq. (23) is true for the index M if it is true for $M - 1$. By induction, therefore, the veracity of Eq. (23) is reduced to the case $M = 0$, $C_0^{(0)} r_0 = r_1$. This last equation is satisfied, being in fact the first equation of Eqs. (17).

4. Realization of Operator—Mathematical Formulation. It is convenient here again to regard the signal as a function of time, $f(t)$. The information which is to be extracted from the signal we again denote by $F(t)$, $F(t)$ being as close as possible to $g(t)$.

We shall consider the nature of a four-terminal linear passive network which, when its input voltage is $f(t)$, has $F(t)$ as its open-circuit output-voltage. The complication inherent in the construction of satisfactory inductive elements makes the use of RC networks common. For this reason and for the sake of simplicity we shall *here restrict ourselves to RC networks.*

We denote by $A(t)$ the open-circuit output voltage of the net-work corresponding to an input voltage which is a unit-step function. We recall the relationship

$$F(t) = \int_0^\infty A'(\tau) f(t - \tau)\, d\tau + A(0) f(t).$$

We also shall require the Laplace transform of $A(t)$. Here we shall denote by $k(p)$ the Laplace transform of $A(t)$ multiplied by p. The function $k(p)$ is a transfer function. It is given by

$$k(p) = A(0) + \int_0^\infty A'(t) e^{-pt}\, dt. \tag{28}$$

In the case of a network free of inductances $k(p)$ is a rational function having its poles on the negative real axis of the complex p-plane. Moreover the poles of $k(p)$ are simple and $k(p)$ is representable by a partial fraction expansion

$$k(p) = \alpha_0 + \sum_{m=1}^{K} \frac{\alpha_m}{p + \sigma_m} \tag{29}$$

where $0 < \sigma_1 < \sigma_2 < \cdots < \sigma_K$ and the α_m are real. From Eq. (28) we see that Eq. (29) is satisfied if $\alpha_0 = A(0)$ and if

$$A'(\tau) = \sum_{m=1}^{K} \alpha_m e^{-\sigma_m \tau} \tag{30}$$

We recall that in Eqs. (2) we used $hA'(nh) = A_n$. Thus we want A_n to be equal to

$$h \sum_{m=1}^{K} \alpha_m e^{-nh\sigma_m}, \qquad n > 0, \tag{31}$$

The A_n, we recall, are already determined as in Sec. 3. Here we are trying to find $k(p)$. In general, to meet the requirement that A_n be given by Eqs. (31) exactly would require a filter of unwarranted complexity. Therefore, setting

$$B_n = h \sum_{m=1}^{K} \alpha_m e^{-nh\sigma_m}, \qquad n > 0, \tag{32}$$

we require that $A_n - B_n$ be small. The extent to which we can make B_n close to A_n depends on how large we make K. If the successive A_n form a slowly changing sequence, K can be chosen much smaller than M. We recall that M is the number of A_n, $n > 0$. On the other hand, if the A_n change markedly from one value of n to the next it may be necessary to take K as large as M. The smaller K can be made, the simpler the filter. A plot of the values of A_n against n should be of great help in deciding how much the A_n fluctuate between successive values of n and therefore how large to take K.

Our problem is really one in approximation. We want to make Eqs. (31) represent A_n as closely as possible. We have at our disposal the choice of the σ_m and the α_m. It is inadvisable to choose σ_{m+1}/σ_m close to one since this introduces extremes in the sizes of the elements of the filter associated with $k(p)$. It is also inadvisable to take σ_K/σ_1 too large since this has no influence on the B_n except for small n.

In order to be able to carry our analysis further we shall now make the assumption

$$\sigma_m = \beta m/Mh \tag{33}$$

where β is a scale factor. A reasonable choice for β is 1. Under certain conditions a value of $\frac{1}{2}$ or 2 may provide a better fit of B_n to A_n for a given value of K. *The choice of σ_m as given in Eq. (33) is arbitrary and a quite different choice may be much more useful under certain conditions.* From here on, however, we shall proceed in the basis of the assumption Eq. (33).

We have now to determine the α_m so that the $B_n - A_n$ are small. The effect of the terms $B_n - A_n$ on V as given in Eqs. (13) appears to be

$$\sum_{m,n=1}^{M} (B_n - A_n)(B_m - A_m) r_{n-m}. \tag{34}$$

This, however, is not complete in the present case because the B_n as given by Eqs. (32) are not zero for $n > M$. To prevent these B_n, $n > M$, from affecting the filter output, $F(t)$, too badly we require

$$\sum_{M+1}^{\infty} B_n^2$$

to be small. Recalling Eqs. (32) we can put this condition in more convenient form by requiring that

$$h \int_{Mh}^{\infty} [A'(\tau)]^2 \, d\tau \tag{35}$$

be small where $A'(\tau)$ is given by Eqs. (30). Combining this with Eq. (34) we choose the α_m so as to minimize

$$J = \sum_{m,n=1}^{M} (B_n - A_n)(B_m - A_m) r_{n-m} + \lambda h \int_{Mh}^{\infty} [A'(\tau)]^2 \, d\tau. \tag{36}$$

The value λ in Eq. (36) represents a positive number that is chosen large if it is important to make the influence of $f(t - \tau)$ on $F(t)$ small for $\tau > Mh$.

To see this, we recall that by Eq. (1)

$$F(t) = \int_0^{Mh} A'(\tau) f(t - \tau \, d\tau + A(0) f(t) + H$$

where

$$H = \int_{Mh}^{\infty} A'(\tau) f(t - \tau) \, d\tau.$$

Choosing λ large emphasizes Expression (35) in J, and thus when the α_m are determined by minimizing J, Term (35) will be small. Making Expression (35) small will make H small and therefore $F(t)$ is determined mainly by $f(t - \tau)$, $0 < t < Mh$. Thus as soon as $t > Mh$, $F(t)$ is largely independent of any aberrations in $f(t)$ that occurred when $t < 0$.

In cases where this transient aspect is of little importance, λ is given a smaller value determined by the size of r_n when n is near M in size.

Once a value of λ is decided upon we again have only the α_m to determine. In terms of α_m we have, using Eqs. (30) in Eqs. (36),

$$J = \sum_{m,n=1}^{M} (B_n - A_n)(B_m - A_m) r_{n-m} + \lambda h \sum_{p,q=1}^{K} \frac{\alpha_p \alpha_q e^{-(\sigma_p + \sigma_q) Mh}}{\sigma_p + \sigma_q}, \tag{37}$$

to minimize J we set $\partial J / \partial \alpha_s = 0$. Finding $\partial J / \partial \alpha_s$ we have

$$\frac{\partial J}{\partial \alpha_s} = 2 \sum_{m,n=1}^{M} (B_n - A_n) r_{n-m} \frac{\partial B_m}{\partial \alpha_s} + 2\lambda h \sum_{p=1}^{K} \frac{\alpha_p e^{-(\sigma_p + \sigma_s) Mh}}{\sigma_p + \sigma_s}.$$

Or setting $\partial J / \partial \alpha_s = 0$ and finding $\partial B_m / \partial \alpha_s$ from Eqs. (32), we have

$$h \sum_{m,n=1}^{M} (B_n - A_n) r_{n-m} e^{-mh\sigma_s} + \gamma h \sum_{p=1}^{K} \frac{\alpha_p e^{-(\sigma_p + \sigma_s) Mh}}{\sigma_p + \sigma_s} = 0, \qquad s = 1, 2, \cdots, K.$$

And again using Eqs. (32) we obtain

$$\sum_{p=1}^{K} \alpha_p \left(h^2 \sum_{n,m=1}^{M} e^{-nh\sigma_p - mh\sigma_s} r_{n-m} + \lambda h \frac{e^{-(\sigma_p+\sigma_s)Mh}}{\sigma_p + \sigma_s} \right) = h \sum_{m,n=0}^{M} A_n r_{n-m} e^{-mh\sigma_s}, \qquad s = 1, \cdots, K. \tag{38}$$

Here we have K equations to determine the K numbers α_p. We can write Eqs. (38) as

$$\sum_{p=1}^{K} C_{s,p} \alpha_p = d_s, \qquad s = 1, \cdots, K, \tag{39}$$

where using Eq. (33),

$$d_s = \sum_{m,n=1}^{M} A_n r_{n-m} e^{-m\beta(s/M)}, \qquad s = 1, \cdots, K,$$

and

$$C_{s,p} = \lambda Mh \frac{e^{-\beta(p+s)}}{(p+s)\beta} + h \sum_{n,m=1}^{M} e^{-\beta(np+ms)/M} r_{n-m}.$$

We have in Eqs. (39) a system of equations for determining the α_m.

Once the α_m are determined we have the problem of choosing a network for which $A'(\tau)$ is given by Eqs. (30), or what is equivalent, $k(p)$, by Eqs. (29).

5. RC Filter. Here we begin by summing up the characteristics of a four-terminal linear passive network with only resistors and condensers for its elements.[2]

We denote $\sigma + j\omega$ by p. Let $Z_{11}(p)$ be the driving-point impedance at the input terminals with the output an open circuit. Similarly $Z_{22}(p)$ is the open-circuit driving-point output-terminal impedance. Finally $Z_{12}(p)$ is the transfer impedance between one end with the other an open circuit. Using the subscripts I and 0 to denote the input and output terminals respectively, we have the well-known relationship,

$$\begin{aligned} V_I &= Z_{11} I_I + Z_{12} I_0 \\ V_0 &= Z_{12} I_I + Z_{22} I_0 \end{aligned} \tag{40}$$

If V_I is a complex number denoting the input voltage at some frequency $\omega/2\pi$ and V_0 is the output voltage on open circuit, then it follows from Eq. (40) that

$$V_0 = V_I \frac{Z_{12}(j\omega)}{Z_{11}(j\omega)}.$$

In terms of p we have

$$V_0(p) = V_I(p) \frac{Z_{12}(p)}{Z_{11}(p)}. \tag{41}$$

[2] E. A. Guillemin, "RC-Coupling Networks," RL Report 43, Oct. 11, 1944.

In the notation of the previous section $V_0(p)$ is the Laplace transform of $F(t)$ and $V_1(p)$ of $f(t)$. We have as the Laplace transform of Eq. (1), $V_0(p) = k(p) \cdot V_1(p)$. Comparing this with Eq. (41) we get

$$\frac{Z_{12}(p)}{Z_{11}(p)} = k(p). \tag{42}$$

We have already determined $k(p)$. In this section we shall determine $Z_{11}(p)$ and $Z_{12}(p)$ so that Eq. (42) is satisfied.

It is necessary and sufficient for Z_{11}, Z_{22}, and Z_{12} to satisfy the following criteria in order to characterize a four-terminal linear passive network free of inductances:

1. $Z_{11}(p)$ [and $Z_{22}(p)$] have simple zeros and poles which lie on the negative real axis in the p-plane. Zeros and poles separate each other. Moreover the smallest pole lies to the right of the smallest zero. For positive p, $Z_{11}(p)$ and $Z_{22}(p)$ are positive.
2. $Z_{12}(p)$ has simple poles and is real for real p.
3. A pole of $Z_{12}(p)$ must also be a pole of $Z_{11}(p)$ and $Z_{22}(p)$. In fact if $p = -\sigma'$ is a pole of Z_{12} with residue α_{12} and if α_{11} and α_{22} are the residues of Z_{11} and Z_{22} at $p = -\sigma'$ then it is necessary for

$$\alpha_{11}\alpha_{22} - \alpha_{12}^2 \geqq 0. \tag{43}$$

We shall choose the poles of Z_{12} coincident with those of Z_{11}. Therefore, we see from Eq. (42) that the zeros of $Z_{12}(p)$ are the zeros of $k(p)$ and the zeros of $Z_{11}(p)$ are the poles of $k(p)$. The zeros of Z_{11} are therefore $-\sigma_k$ where $0 < \sigma_1 < \sigma_2 < \cdots < \sigma_K$. As the poles of Z_{11} and Z_{12} we choose $-\sigma_k'$ where

$$0 < \sigma_1' < \sigma_1 < \sigma_2' < \sigma_2 < \cdots < \sigma_K' < \sigma_K. \tag{44}$$

We can take

$$\sigma_1' = \tfrac{1}{2}\sigma_1, \quad \sigma_2' = \tfrac{1}{2}(\sigma_1 + \sigma_2), \quad \sigma_3' = \tfrac{1}{2}(\sigma_2 + \sigma_3), \text{ etc.}$$

Thus we have

$$Z_{11}(p) = e\,\frac{(p+\sigma_1)(p+\sigma_2)\cdots(p+\sigma_K)}{(p+\sigma_1')(p+\sigma_2')\cdots(p+\sigma_K')}, \tag{45}$$

where e is a positive constant that can be chosen to help make the size of the elements of the network physically reasonable. Using Eq. (42) we see that having determined $Z_{11}(p)$, $Z_{12}(p)$ is given by

$$Z_{12}(p) = k(p)Z_{11}(p). \tag{46}$$

It is convenient to choose $Z_{22}(p) = Z_{11}(p)$, thus making the four-terminal nerwork symmetric. In general, such a choice may cause Eq. (43) to be violated. This can be avoided by reducing $k(p)$ by some constant factor, g, and then compensating for this reduction by any one of several amplifying devices. Equation (43) then becomes

$$\alpha_{11}^2 - g^2\alpha_{12}^2 \geqq 0. \tag{47}$$

Clearly g can be chosen so that this condition is satisfied at all the poles, $-\sigma_b'$, of Z_{12}.

Having chosen

$$Z_{11}(p),\ Z_{12}(p),\ \text{and}\ Z_{22}(p)\ [=Z_{11}(p)],$$

all to within an arbitrary coefficient e, we can now follow Guillemin[3] in finding a four terminal network with the desired characteristics. Briefly in the symmetric network shown in Fig. 1, it is necessary for

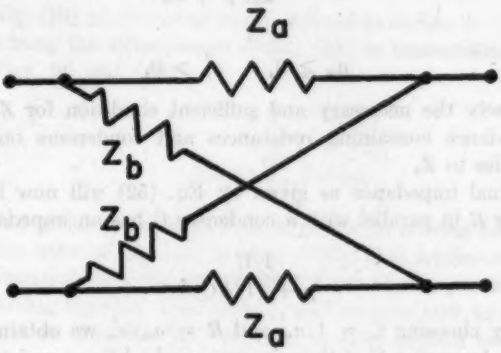

FIG. 1

$$\begin{aligned} Z_a &= Z_{11} - Z_{12} \\ Z_b &= Z_{11} + Z_{12} \end{aligned} \tag{48}$$

in order for the network to have Z_{11}, Z_{12} and Z_{22} as its open-circuit impedances. In this manner the problem is reduced to finding Z_a and Z_b.

The quantities Z_a and Z_b are two-terminal impedances. To see this we observe first that (44) and (45) assure that

$$\alpha_{11},\ m > 0, \tag{49}$$

in the partial fraction expansion

$$Z_{11}(p) = e + \sum_{m=1}^{K} \frac{\alpha_{11,m}}{p + \sigma_m'}. \tag{50}$$

From Eq. (46) we observe that since the poles of $k(p)$ are canceled by the zeros of $Z_{11}(p)$ we have

$$Z_{12}(p) = ek(\infty) + \sum_{m=1}^{K} \frac{\alpha_{12,m}}{p + \sigma_m'}.$$

Using the modified $gk(p)$ in place of $k(p)$ this becomes

$$Z_{12}(p) = egk(\infty) + \sum_{m=1}^{K} \frac{g\alpha_{12,m}}{p + \sigma_m'}. \tag{51}$$

We see from Eq. (28), incidentally, that $k(\infty) = A(0)$.

[3] *Op. cit.*

Using Eqs. (50) and (51) we have

$$Z_a = Z_{11} - Z_{12} = e - egk(\infty) + \sum_{m=1}^{K} \frac{\alpha_{11,m} - g\alpha_{12,m}}{p + \sigma_m'}.$$

By (47) and (49) we see that $\alpha_{11,m} - g\alpha_{12,m} \geqq 0$. Moreover, by further adjusting g if necessary, $e - egk(\infty) \geqq 0$. Thus Z_a is of the form

$$a_0 + \sum_{m=1}^{K} \frac{a_m}{p + \sigma_m'} \tag{52}$$

where

$$a_m \geqq 0, \qquad \sigma_m' > 0,$$

which is precisely the necessary and sufficient condition for Z_a to be a two-terminal impedance containing resistances and condensers only. The same argument applies to Z_b.

A two-terminal impedance as given by Eq. (52) will now be constructed. Since a resistor R in parallel with a condensor C has an impedance

$$\frac{1/C}{p + (1/RC)}.$$

We see that by choosing $C = 1/a_m$ and $R = a_m/\sigma_m'$ we obtain an impedance $a_m/(p + \sigma_m')$. Arranging K of these in series and adding a resistor a_0 we obtain a two-terminal impedance of the desired form. With the choice of a large value of e in Eq. (45) a_m is large and therefore $1/a_m$, the capacitance, can be made reasonably small.

6. **Prediction and Lag With and Without Noise.** In Sec. 2 the problem of separating a message, represented by a sequence a_n, from a signal, represented by a sequence b_n, was considered. The sequence $b_n - a_n$ is called noise. There the optimum set of numbers A_n was determined in order that a_k should be represented as closely as possible by

$$\sum_{n=0}^{M} A_n b_{k-n}. \tag{53}$$

In Eq. (53) we utilize b_k and earlier values such as b_{k-1}, b_{k-2}, etc., in deriving a_k. There are situations where on the basis of knowing b_k, b_{k-1}, b_{k-2}, etc., we must use Eq. (53) to represent not a_k but a_{k+s}, where s is a positive integer. Here we have a problem involving not only filtering, that is the separation of message from noise, but also prediction. In other words, even if there were no noise there would still be the problem of determining a_{k+s} from a knowledge of a_k, a_{k-1}, etc. This problem arises in fire control where it is necessary to point a gun not at where the target is but at where it is likely to be by the time the shell arrives.

Proceeding as in Sec. 2 we now choose the A_n so as to minimize the *rms* of

$$\epsilon_k = a_{k+s} - \sum_{n=0}^{M} A_n b_{k-n}. \tag{54}$$

Instead of Eq. (5) we find

$$I = R_a(0) - 2\sum_{n=0}^{M} A_n R_{ia}(n+s) + \sum_{n,m=0}^{M} A_n A_m R_b(m-n).$$

Minimizing I we obtain

$$\sum_{n=0}^{M} A_n r_{k-n} = \gamma_{k+s}, \qquad k = 0, 1, \cdots, M, \tag{55}$$

in place of Eq. (10) where r_k and γ_k are defined as in Sec. 2.

In determining the effectiveness of Eq. (53) in representing a_{k+s} we get now instead of Eqs. (9) and (10),

$$V = 1 - E_M \tag{56}$$

$$E_M = \sum_{n=0}^{M} A_n \gamma_{n+s}. \tag{57}$$

The method given at the beginning of Sec. 3 for solving two systems each of about half the order of Eq. (11) in place of Eq. (11) applies equally well to Eq. (55). The iteration formulas given in Sec. 3 can also be generalized to cover the case of predicting together with filtering and we now turn to this problem.

In place of Eqs. (55) and (57) we have, in more explicit notation,

$$\sum_{n=0}^{M} A_n(M) r_{k-n} = \gamma_{k+s}, \qquad k = 0, 1, \cdots, M, \tag{58}$$

$$E_M = \sum_{n=0}^{M} A_n^{(M)} \gamma_{n+s}. \tag{59}$$

We observe that the only difference between these equations and Eqs. (14) and (15) is in the index of γ which is now increased by s. Thus the only change in Eqs. (16) to (20) is an increase in the index of γ by the number s. Equations (17) and (18) remain unchanged since they do not contain γ and we rewrite them as before

$$C_0^{(M)}\left(r_0 - \sum_{k=0}^{M-1} C_k^{(M-1)} r_{M-k}\right) = r_{M+1} - \sum_{k=1}^{M} C_{k-1}^{(M-1)} r_k, \tag{60}$$

$$C_k^{(M)} = C_{k-1}^{(M-1)} - C_0^{(M)} C_{M-k}^{(M-1)}, \qquad k = 1, 2, \cdots, M. \tag{61}$$

Equation (19) is modified to

$$A_{M+1}^{(M+1)}\left(r_0 - \sum_{n=0}^{M} C_n^{(M)} r_{M+1-n}\right) = \gamma_{M+1+s} - \sum_{n=0}^{M} A_n^{(M)} r_{M+1-n}, \tag{62}$$

whereas Eq. (20) remains unchanged as

$$A_k^{(M+1)} = A_k^{(M)} - C_k^{(M)} A_{M+1}^{(M+1)}, \qquad k = 0, 1, \cdots, M. \tag{63}$$

In place of Eq. (21) we have

$$E_{M+1} = E_M + A_{M+1}^{(M+1)}\left(\gamma_{M+1+s} - \sum_{n=0}^{M} C_n^{(M)} \gamma_{n+s}\right). \tag{64}$$

It is extremely useful to observe that, by Eqs. (60) and (61), the $C_k^{(M)}$ are independent of the choice of s. It is also helpful to observe that the same combination appears in the bracket in the left-hand side of Eqs. (60) and (62) except for the index M in one case and $M+1$ in the other. The $A_k^{(M)}$ depend of course on s. In some cases the range of prediction is taken as far ahead as is possible without causing E_M to fall below some preassigned value. Under such conditions the $A_k^{(M)}$ and E_M must be recomputed for several choices of s. The fact that $C_k^{(M)}$ is independent of s greatly facilitates the computation.

So far, for the sake of being definite, we have considered the case $s > 0$ and discussed the prediction problem. The case $s < 0$ also is of considerable importance. By taking $s < 0$ it is possible to improve the separation of message from noise. If such improvement turns out to be appreciable and if the lag in obtaining the sequence a_{k+s}, $s < 0$, instead of a_k, is not important, then of course it is worth while to take $s < 0$. *All* the formulas given here are valid for any integer value of s, whether positive or negative.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY.

SEE J. MATH & PHYS. 27, 240, 1948 FOR ERRATA

STRESSES AND SMALL DISPLACEMENTS OF SHALLOW SPHERICAL SHELLS. II

By Eric Reissner

1. Introduction. In the present paper we give some applications to specific problems of the results of an earlier paper on the subject under consideration.[1] We show first that the two simultaneous fourth order equations of the theory of shallow spherical shells may be reduced to two independent second order equations. We then give the solution of the second order equations for the case of rotational symmetry.

With this solution we obtain explicit results for the following three problems, (1) a shell with no edge restraint carrying a point load at the apex; (2) a shell with no edge restraint carrying a load uniformly distributed over a small circular area with center at the apex (Fig. 1); (3) a shell with edge restraint carrying a point load at the apex (Fig. 11).

We also show in which way our solution may be used to solve a problem of pure bending of curved bars.

We have attempted to present the general results of this paper in such a manner that future applications to other conditions of loading and support are facilitated.

2. The equation of the theory for rotationally symmetric bending and stretching. With the notation of Figs. 1 and 2 and indicating differention with respect to r by primes the system of equations for stress resultants, stress couples and components of displacement for the rotationally symmetric case assumes the following form[1]

$$M_{rr} = -D(w'' + \nu w'/r) \tag{1a}$$

$$M_{\theta\theta} = -D(w'/r + \nu w'') \tag{1b}$$

$$V_r = -D(\nabla^2 w)' \tag{1c}$$

$$N_{rr} = F'/r + \Omega \tag{2a}$$

$$N_{\theta\theta} = F'' + \Omega \tag{2b}$$

$$\Omega' = -p_r \tag{2c}$$

$$R_r = V_r - rN_{rr}/R \tag{3}$$

$$\epsilon_r = u' + (w/R) = (N_{rr} - \nu N_{\theta\theta})/tE \tag{4a}$$

$$\epsilon_\theta = (u/r) + (w/R) = (N_{\theta\theta} - \nu N_{rr})/tE = u_h/r \tag{4b}$$

$$\nabla^2\nabla^2 F - (tE/R)\nabla^2 w = -(1-\nu)\nabla^2\Omega \tag{5a}$$

$$D\nabla^2\nabla^2 w + (1/R)\nabla^2 F = p - (2\Omega/R) \tag{5b}$$

$$D = Et^3/12(1-\nu^2) \tag{6a}$$

$$\nabla^2 = d^2/dr^2 + r^{-1}d/dr \tag{6b}$$

[1] J. Math. Phys. **25**, 80–85, (1946).

Equations (5) have to be solved for the normal component of displacement w and for the stress function F. This solution is to be substituted in the remaining equations (1) to (4).

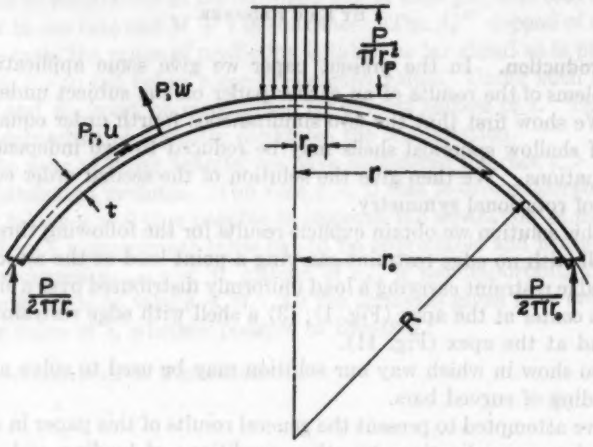

FIG. 1

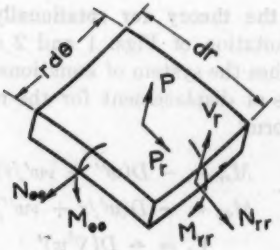

FIG. 2

3. Solution of the homogeneous system of differential equations. Equations (5), with the righthand sides set equal to zero, may be solved as follows. We multiply (5b) by $1/D$ and (5a) by a factor λ and add the resulting equations. This gives

$$\nabla^2\nabla^2(w + \lambda F) - \lambda(tE/R)\nabla^2(w - F/\lambda tED) = 0 \tag{7}$$

Equation (7) becomes an equation for $w + \lambda F$ if we set $\lambda = -1/\lambda tED$ or

$$\lambda = i\sqrt{12(1 - \nu^2)/Et^2} \tag{8}$$

We then define a quantity l by the relation

$$\lambda tE/R = i/l^2 \tag{9}$$

From (8) and (9) follows for l,

$$l = \sqrt{Rt}/\sqrt[4]{12(1-\nu^2)} \tag{10}$$

The differential equation (7) is now of the form

$$\nabla^2\nabla^2(w + \lambda F) - (i/l^2)\nabla^2(w + \lambda F) = 0 \tag{11}$$

and may be integrated to

$$w + \lambda F = \Phi + \Psi \tag{12}$$

where Φ and Ψ are the general solutions of

$$\nabla^2\Phi = 0, \qquad \nabla^2\Psi - (i/l^2)\Psi = 0 \tag{13a, b}$$

The homogeneous system (5) of two simultaneous equations of the fourth order is thus reduced to the solution of two independent equations of the second order. It may be noted that in this reduction no use has been made of the fact that we are dealing with the rotationally symmetric case, so that equation (12) and (13) are valid also for the general problem without rotational symmetry.

Assuming rotational symmetry the solutions of (13) are of the form

$$\Phi = A_1 + A_2 \ln r \tag{14a}$$

$$\Psi = A_3 I_0(\sqrt{i}\, r/l) + A_4 K_0(\sqrt{i}\, r/l) \tag{14b}$$

where the A_n are arbitrary complex constants and I_0 and K_0 are modified Bessel functions of the zero'th order. We write further[2]

$$I_0(\sqrt{i}\, x) = \operatorname{ber} x + i \operatorname{bei} x \tag{15a}$$

$$K_0(\sqrt{i}\, x) = \ker x + i \operatorname{kei} x \tag{15b}$$

Substituting equations (15) and (14) in equation (12), separating real and imaginary parts, we obtain with real constants C_n the following expressions for w and F,

$$w = C_1 \operatorname{ber} r/l + C_2 \operatorname{bei} r/l + C_3 \ker r/l + C_4 \operatorname{kei} r/l + C_5 + C_7 \ln r/l \tag{16a}$$

$$F = (Et^2/\sqrt{12(1-\nu^2)})[C_1 \operatorname{bei} r/l - C_2 \operatorname{ber} r/l + C_3 \operatorname{kei} r/l - C_4 \ker r/l + C_6 \ln r/l + C_8] \tag{16b}$$

There are *six* essential constants of integration in equations (16). C_8 is immaterial as only derivatives of F occur and C_7 must be put equal to zero, as will be shown, in order that there be no displacement in circumferential direction.

[2] See for instance *"A Treatise on Bessel functions and their application to Physics"* by Gray and Mathews, second edition prepared by Gray and MacRobert, 1931, p. 26.

4. The condition of vanishing circumferential displacement component. We combine equations (4) and (2) in the form

$$\begin{aligned} \epsilon_r - (r\epsilon_\theta)' &= [w - (rw)']/R \\ &= (1/tE)[F'/r - \nu F'' - (rF'')' \\ &\qquad + \nu F'' + (1 - \nu)(\Omega - (r\Omega)')] \end{aligned} \tag{17}$$

Equation (17) simplifies to

$$-rw'/R = (1/tE)[F'/r - F'' - rF''' + (1 - \nu)r\Omega'] \tag{18}$$

and may be written in the form

$$(tE/R)w' = (\nabla^2 F + (1 - \nu)\Omega)' \tag{19}$$

From (19) follows

$$(tE/R)w = \nabla^2 F + (1 - \nu)\Omega + \text{const.} \tag{20}$$

Comparison of (20) with (16) shows that w cannot contain a logarithmic term and consequently we must set

$$C_7 = 0 \tag{21}$$

5. Note on the problem of pure bending of curved bars. It may be worthwhile to indicate that the C_7-term may be used to solve the problem of pure bending about an axis perpendicular to the base plane of the shell for an open circular-ring shell. Instead of assuming that the circumferential displacement component v vanishes we set

$$v = kr\theta \tag{22}$$

According to equation (7c) of reference 1 the condition $\gamma_{r\theta} = 0$ remains unchanged. According to equation (7b) of reference 1, we have now for the circumferential normal strain

$$\epsilon_\theta = k + (u/r) + (w/R) \tag{23}$$

while ϵ_r is still given by equation (4a).

Proceeding now as in equations (17) to (19), but with $\Omega = 0$, we obtain

$$tEk/r + (tE/R)w' = (\nabla^2 F)' \tag{24}$$

and from (24)

$$tEk \ln r/l + (tE/R)w = \nabla^2 F + \text{const.} \tag{25}$$

Comparison of (25) and (16) shows that now

$$C_7 = -kR \tag{26}$$

The constants C_1 to C_7 are to be determined by means of the following six conditions; $M_{rr}(r_0) = M_{rr}(r_i) = N_{rr}(r_0) = N_{rr}(r_i) = R_v = 0$ and $t\int_{r_i}^{r_0} r\sigma_\theta dr = M_\theta$. The constant C_8 in the expression for w remains arbitrary.[3]

[3] For the case $R = \infty$ the solution of this problem has been given by H. Golovin (1881). See S. Timoshenko, Theory of Elasticity, pp. 58–62, 1934.

6. Some Properties of the functions ber, bei, ker, kei. Tables of these functions and their derivatives are in existence[4] and have been used in what follows.

For small argument values we have

$$\operatorname{ber} x = 1 - x^4/2^2\cdot 4^2 + \cdots \tag{27a}$$

$$\operatorname{bei} x = x^2/2^2 - x^6/2^2\cdot 4^2\cdot 6^2 + \cdots \tag{27b}$$

$$\ker x = -\ln x + .1159 + \pi x^2/16 + \cdots \tag{27c}$$

$$\operatorname{kei} x = -(x^2/4)\ln x - \pi/4 + 1.1159x^2/4 + \cdots \tag{27d}$$

$$x^{-1}\operatorname{ber}' x = -x^2/16 + \cdots \tag{28a}$$

$$x^{-1}\operatorname{bei}' x = \tfrac{1}{2} - x^4/2^2\cdot 4^2\cdot 6 + \cdots \tag{28b}$$

$$x^{-1}\ker' x = -x^{-2} + \pi/8 + (x^2/16)\ln x + \cdots \tag{28c}$$

$$x^{-1}\operatorname{kei}' x = -\tfrac{1}{2}\ln x - \tfrac{1}{4} + .558 + \cdots. \tag{28d}$$

For large argument values we have the asymptotic relations,

$$\operatorname{ber} x \approx \frac{\exp(x/\sqrt{2})}{\sqrt{2\pi x}}\cos\left(\frac{x}{\sqrt{2}} - \frac{\pi}{8}\right) \tag{29a}$$

$$\operatorname{bei} x \approx \frac{\exp(x/\sqrt{2})}{\sqrt{2\pi x}}\sin\left(\frac{x}{\sqrt{2}} - \frac{\pi}{8}\right) \tag{29b}$$

$$\ker x \approx \frac{\exp(-x/\sqrt{2})}{\sqrt{2x/\pi}}\cos\left(\frac{x}{\sqrt{2}} + \frac{\pi}{8}\right) \tag{29c}$$

$$\operatorname{kei} x \approx \frac{\exp(-x/\sqrt{2})}{\sqrt{2x/\pi}}\sin\left(\frac{x}{\sqrt{2}} - \frac{\pi}{8}\right) \tag{29d}$$

The following formulas express the second derivatives of the four functions in terms of the first derivatives and the functions themselves,

$$\operatorname{ber}'' x = -\operatorname{bei} x - x^{-1}\operatorname{ber}' x \tag{30a}$$

$$\operatorname{bei}'' x = \operatorname{ber} x - x^{-1}\operatorname{bei}' x \tag{30b}$$

$$\ker'' x = -\operatorname{kei} x - x^{-1}\ker' x \tag{30c}$$

$$\operatorname{kei}'' x = \ker x - x^{-1}\operatorname{kei}' x \tag{30d}$$

Finally, we list the following alternate notation which is sometimes used,[5]

$$\operatorname{ber} x = Re J_0(\sqrt{i}x), \qquad \operatorname{bei} x = Im J_0(\sqrt{i}x) \tag{31a, b}$$

$$\ker x = (\pi/2)Re H_0^{(1)}(\sqrt{i}x), \qquad \operatorname{kei} x = (\pi/2)Im H_0^{(1)}(\sqrt{i}x) \tag{31c, d}$$

$$\ker' x = (\pi/2\sqrt{2})(Im H_1^{(1)} + Re H_1^{(1)}) \tag{31e}$$

$$\operatorname{kei}' x = (\pi/2\sqrt{2})(Im H_1^{(1)} - Re H_1^{(1)}) \tag{31f}$$

[4] H. B. Dwight Tables of integrals and other mathematical data. New York 1934.

[5] See H. Bateman and R. C. Archibald, *Guide to Tables of Bessel Functions* in Math. Tables and Aids to Comp. *1*, p. 253, (1944).

7. Particular integrals for certain kinds of loading. Having the general solution (16) of the differential equations (5) with righthand sides equal to zero one can always find appropriate particular integrals by the method of variation of parameters. It is more convenient for certain kinds of load functions which are of interest to proceed in a more direct manner, as follows.

(a) *Uniform normal load* ($p = p_0$, $\Omega = 0$). One form of the solution is as in the membrane theory of shells,

$$F_p = \tfrac{1}{4} p_0 R r^2, \qquad w_p = 0 \tag{32a, b}$$

A second form of the particular solution may be obtained by assuming power series for F_p and w_p. This second form has the advantage of reducing directly to the corresponding solution of flat-plate theory in the limiting case $R = \infty$. One finds by the method of undetermined coefficients

$$w_p = \frac{p_0 r^4}{16D} \sum_{n=0}^{\infty} \frac{(-1)^n (r/2l)^{4n}}{[(2n+2)!]^2}$$

$$= \frac{p_0 r^4}{16D} \left\{ \tfrac{1}{4} - \frac{1}{(4!)^2} \left(\frac{r}{2l}\right)^4 + \frac{1}{(6!)^2} \left(\frac{r}{2l}\right)^8 - \cdots \right\} \tag{33a}$$

$$F_p = p_0 R l^2 \sum_{n=0}^{\infty} \frac{(-1)^n (r/2l)^{6+4n}}{[(2n+3)!]^2}$$

$$= p_0 R l^2 \left\{ \frac{1}{(3!)^2} \left(\frac{r}{2l}\right)^6 - \frac{1}{(5!)^2} \left(\frac{r}{2l}\right)^{10} + \cdots \right\} \tag{33b}$$

The series will converge rapidly for sufficiently small values of r/l.

It is apparent that the same series method may be used to obtain corresponding particular integrals when the load function p is of the form $p_m r^m$.

(b) *Parabolic normal load* ($p = p_2 r^2$, $\Omega = 0$). The particular integral corresponding to membrane theory is of the form

$$F_p = p_2 R r^4/16, \qquad w_p = p_2 R^2 r^2/Et \tag{34a, b}$$

(c) *Inertia load of a rotating shell.* Let ω be the angular velocity and ρ the mass per unit area of the shell. Then

$$p_r = \omega^2 \rho r, \qquad p = \omega^2 \rho r^2/R, \tag{35a, b}$$

and

$$\Omega = -\tfrac{1}{2}\omega^2 \rho r^2, \qquad (1-\nu)\nabla^2 \Omega = -2(1-\nu)\omega^2 \rho, \tag{36a, b}$$

$$p - 2\Omega/R = 2\omega^2 \rho r^2/R. \tag{36c}$$

Substituting equations (36b) and (36c) in equations (5) there is obtained

$$F_p = \omega^2 \rho r^4/8, \qquad w_p = \tfrac{1}{2}(3+\nu) R \omega^2 \rho r^2/Et \tag{37a, b}$$

An alternate form of the solution which reduces to the solution for the flat plate when $R = \infty$ may again be obtained in the form of power series for F_p and w_p.

8. Expressions for stress resultants and couples. We incorporate in these expressions the particular integrals corresponding to a uniform normal load, according to equation (32) and obtain on the basis of equations (16) and (21),

$$N_{rr} = \frac{Et^2}{\sqrt{12(1-\nu^2)}\,l^2}\left\{C_1 \operatorname{bei}'\frac{r}{l} - C_2 \operatorname{ber}'\frac{r}{l} + C_3 \operatorname{kei}'\frac{r}{l} - C_4 \operatorname{ker}'\frac{r}{l} + C_5\left(\frac{r}{l}\right)^{-1}\right\}\left(\frac{r}{l}\right)^{-1} + \tfrac{1}{2}p_0R \tag{38}$$

$$N_{\theta\theta} = \frac{Et^2}{\sqrt{12(1-\nu^2)}\,l^2}\left\{C_1 \operatorname{bei}''\frac{r}{l} - C_2 \operatorname{ber}''\frac{r}{l} + C_3 \operatorname{kei}''\frac{r}{l} - C_4 \operatorname{ker}''\frac{r}{l} - C_5\left(\frac{r}{l}\right)^{-2}\right\} + \tfrac{1}{2}p_0R \tag{39}$$

$$V_r = -Dl^{-3}\{-C_1 \operatorname{bei}' r/l + C_2 \operatorname{ber}' r/l - C_3 \operatorname{kei}' r/l + C_4 \operatorname{ker}' r/l\} \tag{40}$$

$$R_r = V_r - rN_{rr}/R = -\frac{Et^2}{R\sqrt{12(1-\nu^2)}}\frac{C_5}{r} - \tfrac{1}{2}p_0R \tag{41}$$

$$M_{rr} = -\frac{D}{l^2}\left\{C_1\left[\operatorname{ber}\frac{r}{l} - (1-\nu)\frac{\operatorname{bei}' r/l}{r/l}\right] + C_2\left[\operatorname{bei}\frac{r}{l} + (1-\nu)\frac{\operatorname{ber}' r/l}{r/l}\right] + C_3\left[\operatorname{ker}\frac{r}{l} - (1-\nu)\frac{\operatorname{kei}' r/l}{r/l}\right] + C_4\left[\operatorname{kei}\frac{r}{l} + (1-\nu)\frac{\operatorname{ker}' r/l}{r/l}\right]\right\} \tag{42}$$

$$M_{\theta\theta} = -\frac{D}{l^2}\left\{C_1\left[\frac{\operatorname{bei}' r/l}{r/l} - (1-\nu)\operatorname{ber}\frac{r}{l}\right] - C_2\left[\frac{\operatorname{ber}' r/l}{r/l} + (1-\nu)\operatorname{bei}\frac{r}{l}\right] + C_3\left[\frac{\operatorname{kei}' r/l}{r/l} - (1-\nu)\operatorname{ker}\frac{r}{l}\right] - C_4\left[\frac{\operatorname{ker}' r/l}{r/l} + (1-\nu)\operatorname{kei}\frac{r}{l}\right]\right\}. \tag{43}$$

For external loads different from the uniform normal load distribution it is only necessary to calculate the additional resultants and couples which correspond to the additional solutions F_p and w_p.

9. Solution for a shell with point load $-P$ at the apex and with no edge restraint. We have to determine the six constants C_n in equations (16) and (38)

to (43) by means of the following conditions

$$R_v = P/2\pi r \tag{44a}$$

$$r = 0;\ w, w', N_{rr}, N_{\theta\theta} \text{ finite} \tag{44b}$$

$$r = \infty;\ w = M_{rr} = M_{\theta\theta} = 0 \tag{44c}$$

Comparison of (41) and (44a) gives

$$C_6 = -(P/2\pi)R\sqrt{12(1-\nu^2)}/Et^3 \tag{45}$$

Comparison of (16a), (38), (39) and (44b) gives, in view of (27c) and (28c),

$$C_3 = 0, \quad C_4 + C_6 = 0 \tag{46}$$

Comparison of (16a), (42), (43) and (44c) gives, in view of (29),

$$C_1 = C_2 = C_5 = 0 \tag{47}$$

Introducing (45) to (47) and (21) into (16) we have for the normal displacement w and for the stress function F,

$$w = \frac{\sqrt{12(1-\nu^2)}}{2\pi}\frac{PR}{Et^2}\,\mathrm{kei}\,\frac{r}{l} \tag{48a}$$

$$F = -\frac{PR}{2\pi}\left(\ker\frac{r}{l} + \ln\frac{r}{\varrho}\right) \tag{48b}$$

From (48a) and (27a) follows for the deflection at the point of load application

$$w(0) = -\tfrac{1}{4}\sqrt{3(1-\nu^2)}\,PR/Et^2 \tag{49}$$

The direct stresses $\sigma_D = N/t$ are obtained from (39) and (40) in the form

$$\sigma_{r,D} = -\frac{\sqrt{12(1-\nu^2)}}{2\pi}\frac{P}{t^2}\left[\frac{1}{(r/l^2)} + \frac{\ker' r/l}{r/l}\right] \tag{50a}$$

$$\sigma_{\theta,D} = -\frac{\sqrt{12(1-\nu^2)}}{2\pi}\frac{P}{t^2}\left[\frac{-1}{(r/l)^2} - \frac{\ker' r/l}{r/l} - \mathrm{kei}\,\frac{r}{l}\right] \tag{50b}$$

From (50), (27) and (28) follows for the direct stresses at the point of load application

$$\sigma_{r,D}(0) = \sigma_{\theta,D}(0) = -\tfrac{1}{8}\sqrt{3(1-\nu^2)}P/t^2 \tag{51}$$

Finally, the bending stresses $\sigma_B = \pm 6M/t^2$ follow from (42) and (43) in the form

$$\sigma_{r,B} = \mp\frac{3P}{\pi t^2}\left[\ker\frac{r}{l} - \frac{(1-\nu)}{r/l}\,\mathrm{kei}'\frac{r}{l}\right] \tag{52a}$$

$$\sigma_{\theta,B} = \mp\frac{3P}{\pi t^2}\left[\frac{1-\nu}{r/l}\,\mathrm{kei}'\frac{r}{l} + \nu\ker\frac{r}{l}\right] \tag{52b}$$

Equations (52) reduce for small values of r/l to

$$\sigma_{r,B} \approx \sigma_{\theta,B} \approx \pm\tfrac{1}{2}(1+\nu)\frac{3P}{\pi t^2}\ln\frac{r}{l} \tag{53}$$

Fig. 3 contains graphs of the direct stress distribution and of the bending stress distribution on the outer face of the shell. These have been calculated by means of equations (50) and (52), with a value of Poisson's ratio of 0.20.

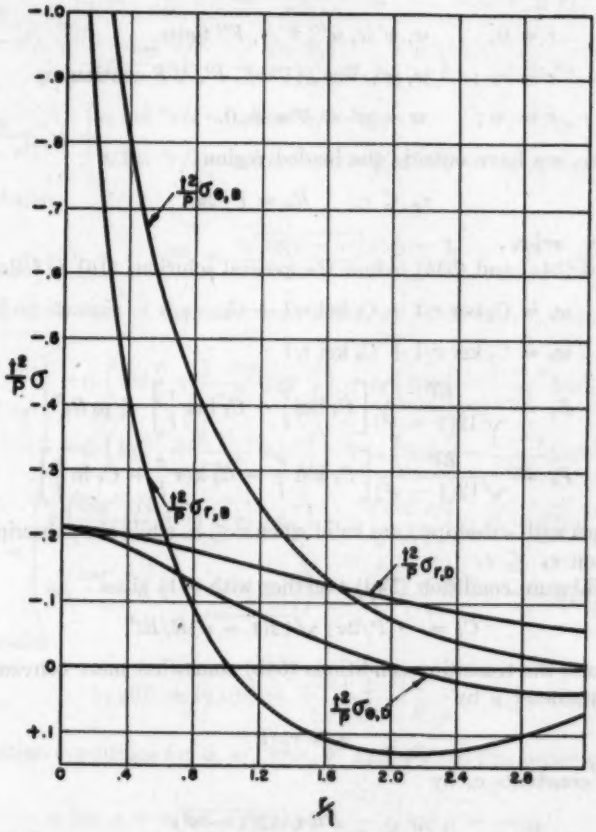

FIG. 3. Direct stresses and bending stresses on outer face of shell of thickness t and radius R caused by a point load $-P$ at the apex of the shell ($l = \sqrt{Rt}/\sqrt[4]{12(1-\nu^2)}$; $\nu = .20$).

Figure 10 contains a plot of the deflection w calculated according to equation (48).[6]

[6] For some of the formulas given in this section see also a paper by J. W. Geckeler in Ingenieur Archiv. *1*, 255-270, (1930). Geckeler's analysis, which is quite different from ours, suffers from the fact that he obtains a formula for the deflection w which contains a term which becomes logarithmically infinite at the point of load application. He neglects this infinite term compared with the terms which remain finite and arrives in this way at equation (48a).

10. Solution for a shell with load $-P$ uniformly distributed over a small circular area with center at apex. No edge restraint. Designating the radius of the load area by r_P we have the following boundary and transition conditions

$$r = 0; \qquad w, w'/r, w'', F'/r, F''' \text{ finite} \tag{54a}$$

$$r = r_P; \qquad w, w', \nabla^2 w, (\nabla^2 w)', F', \nabla^2 F \text{ continuous} \tag{54b}$$

$$r = \infty; \qquad w = w' = \nabla^2 w = 0. \tag{54c}$$

Furthermore, we have outside the loaded region

$$r_P \leq r; \qquad R_v = P/2\pi r, \tag{54d}$$

where $P = \pi r_P^2 p_0$.

Equations (54a) and (54c) reduce the general solutions (16) as follows,

$$w_i = C_1 \operatorname{ber} r/l + C_2 \operatorname{bei} r/l + C_3 \tag{55a}$$

$$w_0 = C_3 \operatorname{ker} r/l + C_4 \operatorname{kei} r/l \tag{55b}$$

$$F_i = \frac{Et^2}{\sqrt{12(1-\nu^2)}}\left[C_1 \operatorname{bei}\frac{r}{l} - C_2 \operatorname{ber}\frac{r}{l}\right] - \tfrac{1}{4} p_0 R r^2 \tag{56a}$$

$$F_0 = \frac{Et^2}{\sqrt{12(1-\nu^2)}}\left[C_3 \operatorname{kei}\frac{r}{l} - C_4 \operatorname{ker}\frac{r}{l} + C_5 \ln\frac{r}{l}\right]. \tag{56b}$$

The solutions with subscript i are valid when $r \leq r_P$ while the subscript 0 refers to the region $r_P \leq r$.

The equilibrium condition (54d) together with (41) gives

$$C_5 = -(P/2\pi)\sqrt{12(1-\nu^2)}R/Et^2 \tag{57}$$

To evaluate the transition conditions (54b) somewhat more conveniently we define a parameter μ by

$$\mu = r_P/l \tag{58}$$

and define constants c_n by

$$C_n = c_n \frac{PR\sqrt{12(1-\nu^2)}}{\pi E t^2 \mu^2}. \tag{59}$$

We introduce (59) and (58) into (55) and (56) and obtain in terms of the constants C_n

$$\pi E t^2 \mu^2 w_i = PR\sqrt{12(1-\nu^2)}[c_1 \operatorname{ber} r/l + c_2 \operatorname{bei} r/l + c_3] \tag{60a}$$

$$\pi E t^2 \mu^2 w_0 = PR\sqrt{12(1-\nu^2)}\,[c_3 \operatorname{ker} r/l + c_4 \operatorname{kei} r/l] \tag{60b}$$

In particular,

$$w_i(0) = \frac{PR}{Et^2}\frac{\sqrt{12(1-\nu^2)}}{\pi}\frac{c_1 + c_3}{\mu^2} \tag{60c}$$

The direct stresses $\sigma_D = N/t$ become

$$\frac{t^2\pi\mu^2\sigma_{r,D}}{P\sqrt{12(1-\nu^2)}} = \begin{cases} c_1\dfrac{\mathrm{bei}'\,r/l}{r/l} - c_2\dfrac{\mathrm{ber}'\,r/l}{r/l} - \frac{1}{2}, & r \le r_p \quad (61a) \\ c_3\dfrac{\mathrm{kei}'\,r/l}{r/l} - c_4\dfrac{\mathrm{ker}'\,r/l}{r/l} - \frac{1}{2}\left(\dfrac{\mu}{r/l}\right)^2, & r \le r_p \quad (61b) \end{cases}$$

$$\frac{t^2\pi\mu^2\sigma_{\theta,D}}{P\sqrt{12(r-\nu^2)}} = \begin{cases} c_1\,\mathrm{bei}''\,r/l - c_2\,\mathrm{ber}''\,r/l - \frac{1}{2}, & r \le r_p \quad (62a) \\ c_3\,\mathrm{kei}''\,r/l - c_4\,\mathrm{ker}''\,r/l + \frac{1}{2}(\mu l/r)^2, & r_p \le r \quad (62b) \end{cases}$$

and in particular

$$\sigma_{r,D}(0) = \sigma_{\theta,D}(0) = \frac{\sqrt{12(1-\nu^2)}}{2\pi}\frac{P}{t^2}\frac{c_1-1}{\mu^2} \qquad (61c), (62c)$$

The bending stresses of the outermost fibers, $\sigma_B = \pm 6M/t^2$, become

$$\mp\frac{2\pi\mu^2 t^2\sigma_{r,B}}{12P} = \begin{cases} -c_1\left(\mathrm{bei}\,\dfrac{r}{l} + \dfrac{1-\nu}{r/l}\,\mathrm{ber}'\,\dfrac{r}{l}\right) + c_2\left(\mathrm{ber}\,\dfrac{r}{l} - \dfrac{1-\nu}{r/l}\,\mathrm{bei}'\,\dfrac{r}{l}\right) & (63a) \\ -c_3\left(\mathrm{kei}\,\dfrac{r}{l} + \dfrac{1-\nu}{r/l}\,\mathrm{ker}'\,\dfrac{r}{l}\right) + c_4\left(\mathrm{ker}\,\dfrac{r}{l} - \dfrac{1-\nu}{r/l}\,\mathrm{kei}'\,\dfrac{r}{l}\right) & (63b) \end{cases}$$

$$\mp\frac{2\pi\mu^2 t^2\sigma_{\theta,B}}{12P} = \begin{cases} c_1\left(\dfrac{\mathrm{ber}'\,r/l}{r/l} + (1-\nu)\,\mathrm{bei}\,\dfrac{r}{l}\right) + c_2\left(\dfrac{\mathrm{bei}'\,r/l}{r/l} - (1-\nu)\,\mathrm{ber}\,\dfrac{r}{l}\right) & (64a) \\ c_3\left(\dfrac{\mathrm{ker}'\,r/l}{r/l} + (1-\nu)\,\mathrm{kei}\,\dfrac{r}{l}\right) + c_4\left(\dfrac{\mathrm{kei}'\,r/l}{r/l} - (1-\nu)\,\mathrm{ker}\,\dfrac{r}{l}\right) & (64b) \end{cases}$$

and in particular

$$\sigma_{r,B}(0) \pm \sigma_{\theta,B}(0) = \mp\frac{3(1+\nu)}{\pi}\frac{P}{t^2}\frac{c_2}{\mu_2}. \qquad (63c), (64c)$$

The transition conditions for w, w', $\nabla^2 w$, F' and $\nabla^2 F$, in this order, are of the form

$$c_1\,\mathrm{ber}\,\mu + c_2\,\mathrm{bei}\,\mu - c_3\,\mathrm{ker}\,\mu - c_4\,\mathrm{kei}\,\mu = -c_5 \qquad (65a)$$

$$c_1\,\mathrm{ber}'\,\mu + c_2\,\mathrm{bei}'\,\mu - c_3\,\mathrm{ker}'\,\mu - c_4\,\mathrm{kei}'\,\mu = 0 \qquad (65b)$$

$$-c_1\,\mathrm{bei}\,\mu + c_2\,\mathrm{ber}\,\mu + c_3\,\mathrm{kei}\,\mu - c_4\,\mathrm{ker}\,\mu = 0 \qquad (65c)$$

$$c_1\,\mathrm{bei}'\,\mu - c_2\,\mathrm{ber}'\,\mu - c_3\,\mathrm{kei}'\,\mu + c_4\,\mathrm{ker}'\,\mu = 0 \qquad (65d)$$

$$c_1\,\mathrm{ber}\,\mu + c_2\,\mathrm{bei}\,\mu - c_3\,\mathrm{ker}\,\mu - c_4\,\mathrm{kei}\,\mu = 1 \qquad (65e)$$

Comparison of (65a) and (65e) gives the value of c_5 directly as

$$c_5 = -1 \qquad (66)$$

We note that with this value of c_5 in equation (60c) the ratio of center deflection to direct stress at center as given by (61c) is seen to have a value which is

independent of the radius r_p of the load area,

$$w(0)/\sigma_D(0) = 2R/E \tag{67}$$

It remains to solve the four simultaneous equations (65b) to (65e) for c_1, c_2, c_3 and c_4. It happens that the system (65) can be solved in closed form[7] as follows

$$c_1 = -\mu \ker' \mu, \qquad c_2 = \mu \operatorname{kei} \mu, \qquad c_3 = -\mu \operatorname{ber}' \mu, \qquad c_4 = \mu \operatorname{bei}' \mu \tag{68}$$

Some numerical values of these coefficients are given below[8]

$\mu =$	.257,	.515,	.823
$c_1 =$	.975,	.905,	.779
$c_2 =$	.0655,	.1728,	.2956
$c_3 =$	.00028,	.00438,	.02871
$c_4 =$	.0331,	.1323,	.3381

Figs. 4 to 10 contain the results of the calculation of deflections, direct stresses, bending stresses and combined stresses. It is seen that deflections and direct stresses vary only slightly with the radius r_p of the load area while the values of the bending stresses are greatly dependent on the magnitude of r_p.

We may list once more the explicit expressions for stresses and displacement at the apex of the shell, making use of (68),

$$w(0) = -\frac{\sqrt{12(1-\nu^2)}}{\pi}\frac{PR}{Et^2}\left(\frac{\ker' \mu}{\mu} + \frac{1}{\mu^2}\right) \tag{69a}$$

$$\sigma_D(0) = -\frac{\sqrt{12(1-\nu^2)}}{2\pi}\frac{P}{t^2}\left(\frac{\ker' \mu}{\mu} + \frac{1}{\mu^2}\right) \tag{69b}$$

$$\sigma_B(0) = \mp\frac{3(1+\nu)}{\pi}\frac{P}{t^2}\frac{\operatorname{kei} \mu}{\mu} \tag{69c}$$

where according to (10) and (58)

$$\mu = \sqrt[4]{12(1-\nu^2)}\, r_p/\sqrt{Rt}. \tag{69d}$$

11. **Solution for a shell with concentrated load at apex and with edge restrained against rotation.** We assume that the shell is supported by means of a ring of cross-sectional area A_0 and modulus of elasticity E_0, the ring being restrained against rotation (Fig. 11a). The conditions of vertical equilibrium and of finite deflection and direct stresses at the apex are the same as for the

[7] H. G. Savidge, Report of the British Association for the Advancement of Science (1916), page 122.

[8] For a shell with radius $R = 102$ in. and thickness $t = \frac{1}{4}$ in. and when $\nu = .2$ these values of μ correspond to load areas πr_p^2 of 1.5625, 6.25, and 16 in.^{2}, respectively.

shell without edge restraint and are given by equations (44a) and 44b). The conditions of edge support are

$$w(r_0) = w'(r_0) = 0 \tag{70a, b}$$

$$N_{rr}(r_0) = -(A_0E_0/r_0^2)u_h(r_0) \tag{70c}$$

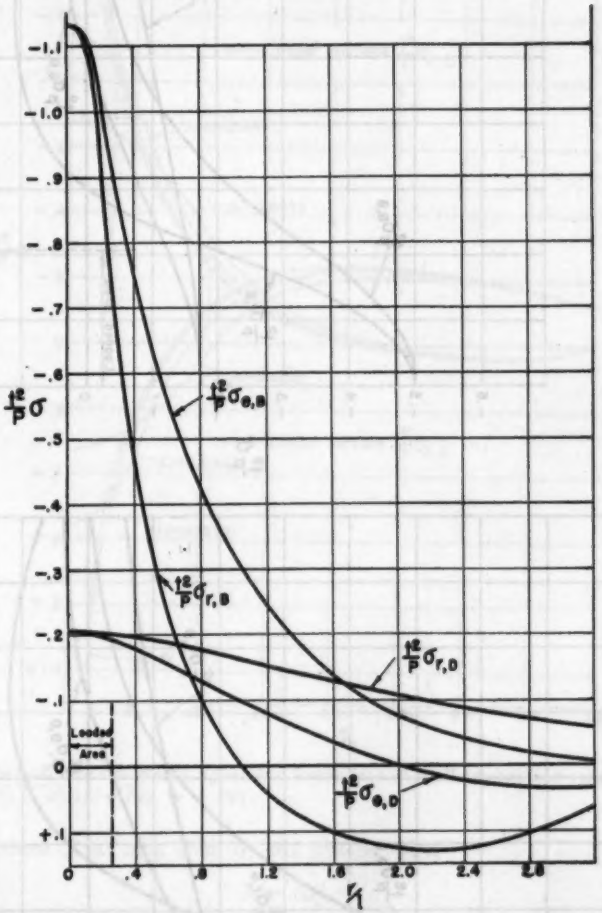

FIG. 4. Direct stresses and bending stresses on outer face of shell caused by load $-P$ uniformly distributed over circular area of radius $r_P = .257l$, ($\nu = .20$).

To satisfy (70) we take w and F instead of as in (48) in the form

$$w = \frac{\sqrt{12(1-\nu^2)}}{2\pi}\frac{PR}{Et^2}\left\{\text{kei}\,\frac{r}{l} + C_1\,\text{ber}\,\frac{r}{l} + C_2\,\text{bei}\,\frac{r}{l} + C_3\right\} \tag{71a}$$

$$F = -\frac{PR}{2\pi}\left\{\text{ker}\,\frac{r}{l} + \ln\frac{r}{l} - C_1\,\text{bei}\,\frac{r}{l} + C_2\,\text{ber}\,\frac{r}{l}\right\}. \tag{71b}$$

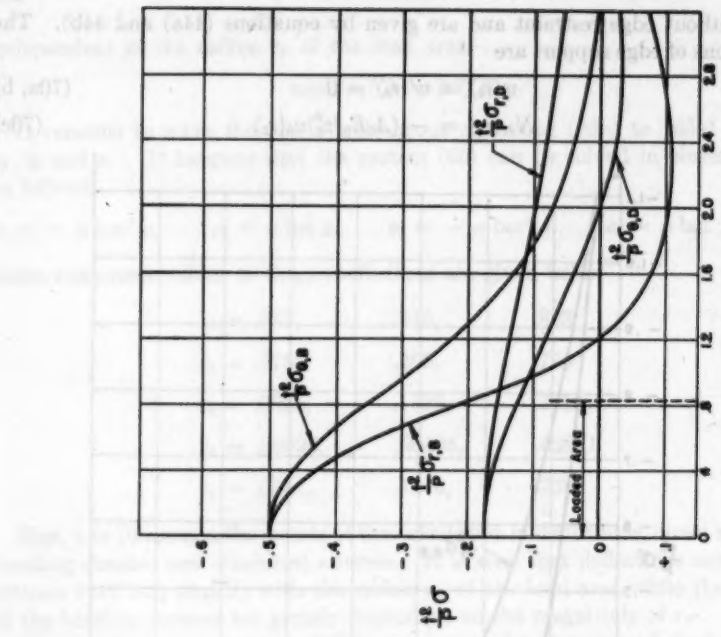

FIG. 6. Same as Fig. 4 for $r_p = .823l$

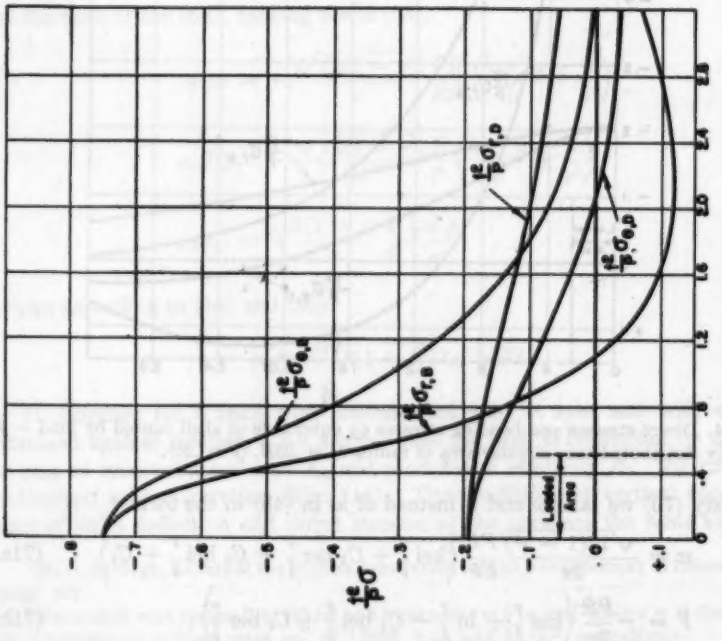

FIG. 5. Same as Fig. 4 for $r_p = .515l$

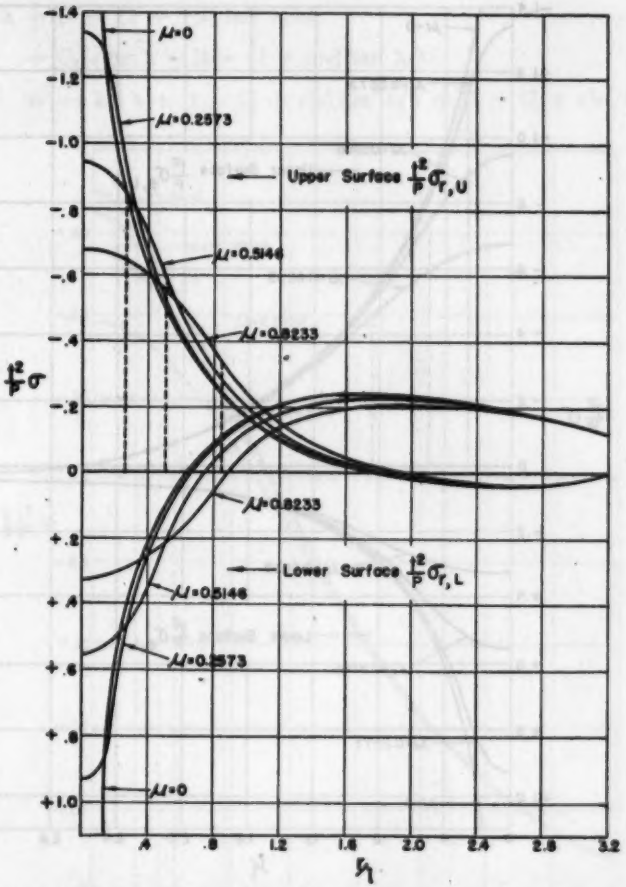

FIG. 7. Meriodional stresses at upper and lower face of shell for various values of $\mu = r_p/l = \sqrt[4]{12(1-\nu^2)}(r_p/\sqrt{Rl})$, ($\nu = .20$).

We introduce (71a) into (70a, b) and obtain, with

$$\lambda = r_0/l, \tag{72}$$

the following two equations for the determination of the constants of integration

$$C_1 \operatorname{ber} \lambda + C_2 \operatorname{bei} \lambda + C_3 = -\operatorname{kei} \lambda \tag{73a}$$

$$C_1 \operatorname{ber}' \lambda + C_2 \operatorname{bei}' \lambda = -\operatorname{kei}' \lambda \tag{73b}$$

To evaluate (70c) we take u_A from (4b) and obtain first, with

$$\kappa = A_0 E_0 / r_0 l E, \tag{74}$$

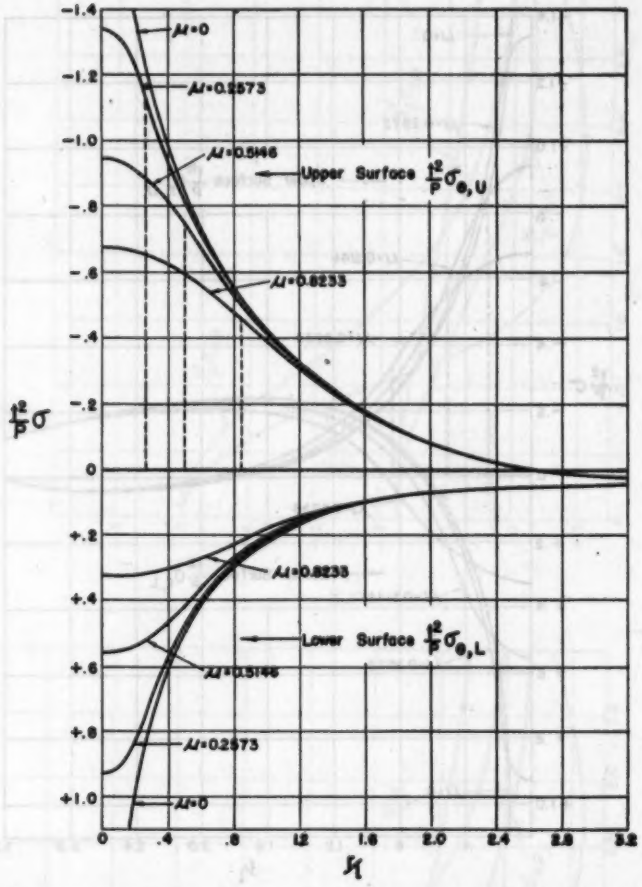

FIG. 8. Circumferential stresses at upper and lower face of shell for various values of $\mu = r_F/l$, ($\nu = .20$).

as third boundary condition

$$(1 - \nu\kappa)N_{rr}(r_0) + \kappa N_{\theta\theta}(r_0) = 0. \tag{70d}$$

Taking N_{rr} and $N_{\theta\theta}$ from (2) we have finally

$$[1 - (1 + \nu\kappa)]F'(r_0)/r_0 + \kappa(\nabla^2 F)_{r_0} = 0. \tag{70e}$$

We introduce (71b) into (70e) and obtain as the third equation connecting the constants C_1, C_2, and C_3,

$$C_1\{\kappa \operatorname{ber}\lambda + [1-(1+\nu)\kappa]\operatorname{bei}'\lambda/\lambda\} + C_2\{\kappa \operatorname{bei}\lambda - [1-(1+\nu)\kappa]\operatorname{ber}'\lambda/\lambda\} = -\kappa \operatorname{kei}\lambda + [1-(1+\nu)\kappa]\operatorname{ker}'\lambda/\lambda + [1-(1+\nu)\kappa]/\lambda^2 \quad (73)$$

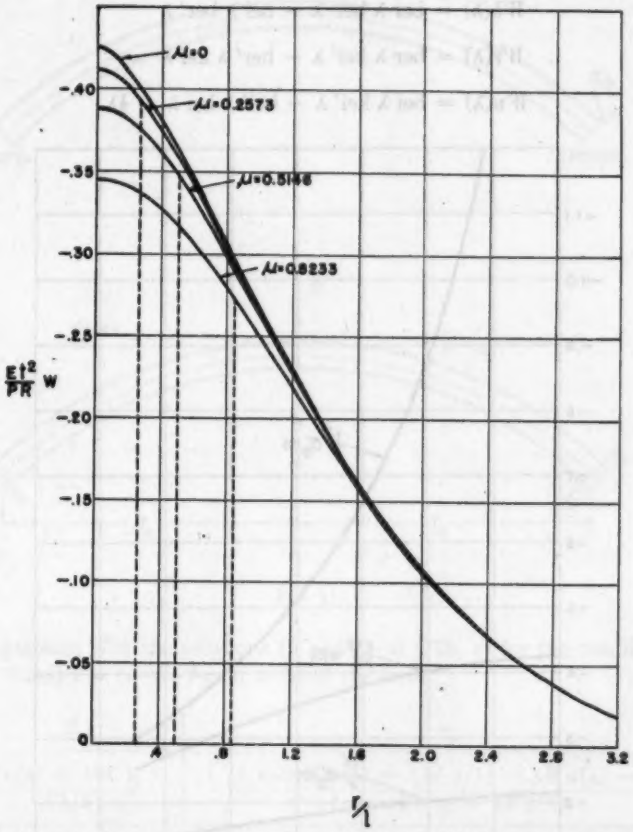

FIG. 9. Normal displacements w of the points of the middle surface of the shell for various values of $\mu = r_P/l$.

In what follows we consider the two limiting cases $\kappa = \infty$ (rigid ring) and $\kappa = 0$ (ring non-resistant to axial forces), thereby obtaining limits for the effect of a ring with finite value of κ.

It is convenient to make use of the following abbreviations for certain product functions which occur in the solution.[9]

[9] H. G. Savidge, loc. cit. p. 109–111.

$$Vb(\lambda) = (\mathrm{ber}'\,\lambda)^2 + (\mathrm{bei}'\,\lambda)^2 \tag{75a}$$

$$Vr(\lambda) = \mathrm{ber}'\,\lambda\,\mathrm{ker}'\,\lambda + \mathrm{bei}'\,\lambda\,\mathrm{kei}'\,\lambda \tag{75b}$$

$$Vu(\lambda) = \mathrm{bei}'\,\lambda\,\mathrm{ker}'\,\lambda - \mathrm{ber}'\,\lambda\,\mathrm{kei}'\,\lambda \tag{75c}$$

$$Wb(\lambda) = \mathrm{ber}\,\lambda\,\mathrm{bei}'\,\lambda - \mathrm{bei}\,\lambda\,\mathrm{ber}'\,\lambda \tag{75d}$$

$$Wr(\lambda) = \mathrm{ber}\,\lambda\,\mathrm{kei}'\,\lambda - \mathrm{ber}'\,\lambda\,\mathrm{kei}\,\lambda \tag{75e}$$

$$Wu(\lambda) = \mathrm{bei}\,\lambda\,\mathrm{kei}'\,\lambda - \mathrm{bei}'\,\lambda\,\mathrm{kei}\,\lambda - \tfrac{1}{2}\lambda \tag{75f}$$

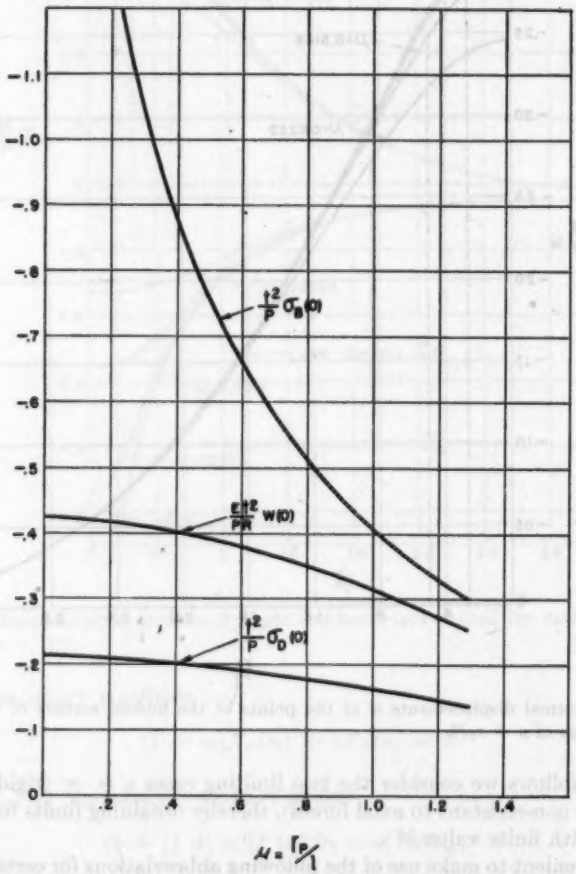

FIG. 10. Stresses and deflection at the apex of the shell for a load $-P$ distributed uniformly over a circular area of radius r_P, as a function of $\mu = \sqrt[4]{12(1-\nu^2)}(r_P/\sqrt{Rt})$.

The first four of these functions are tabulated.[9] Some values of Wr and Wu have been calculated for the present purposes.

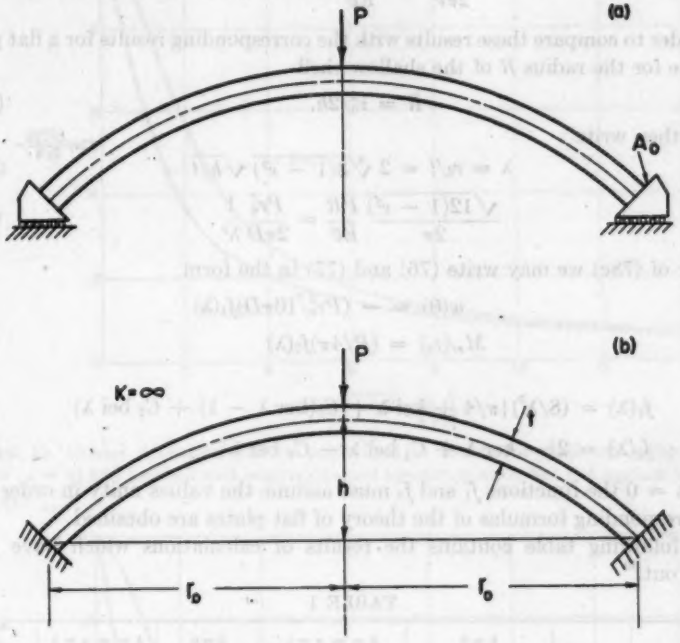

FIG. 11

With equations (75) the solutions C_1 and C_2 of (73b, c) for the two limiting cases $\kappa = 0$ and $\kappa = \infty$ are found to be of the form

κ	0	∞
C_1	$\dfrac{Vu(\lambda) + \text{bei}'\,\lambda/\lambda}{Vb(\lambda)}$	$\dfrac{(1+\nu)[Vu(\lambda) + \text{bei}'\,\lambda/\lambda] - \lambda Wu(\lambda) - \frac{1}{2}}{(1+\nu)Vb(\lambda) - \lambda Wb(\lambda)}$
$-C_2$	$\dfrac{Vr(\lambda) + \text{ber}'\,\lambda/\lambda}{Vb(\lambda)}$	$\dfrac{(1+\nu)[Vr(\lambda) + \text{ber}'\,\lambda/\lambda] - \lambda Wr(\lambda)}{(1+\nu)Vb(\lambda) - \lambda Wb(\lambda)}$

With these values of C_1 and C_2 and with C_3 from (73a) the solution (71) is determined. We shall discuss further (i) the dependence of the center deflection $w(0)$ on λ, (ii) the dependence of the edge bending moment $M_r(r_0)$ on λ.

We find

$$w(0) = -\frac{\sqrt{12(1-\nu^2)}}{2\pi}\frac{PR}{Et^2}\left\{\frac{\pi}{4} + \text{kei}\,\lambda + C_1(1 - \text{ber}\,\lambda) - C_2\,\text{bei}\,\lambda\right\} \quad (76)$$

and

$$M_{rr}(r_0) = -D\frac{\sqrt{12(1-\nu^2)}}{2\pi l^2}\frac{PR}{Eh^2}\{\ker\lambda - C_1\,\mathrm{bei}\,\lambda + C_2\,\mathrm{ber}\,\lambda\}. \quad (77)$$

In order to compare these results with the corresponding results for a flat plate we write for the radius R of the shallow shell

$$R = r_0^2/2h. \quad (78a)$$

We further write

$$\lambda = r_0/l = 2\sqrt[4]{3(1-\nu^2)}\sqrt{h/l} \quad (78b)$$

$$\frac{\sqrt{12(1-\nu^2)}}{2\pi}\frac{PR}{Eh^2} = \frac{Pr_0^2}{2\pi D}\frac{1}{\lambda^2}. \quad (78c)$$

In view of (78c) we may write (76) and (77) in the form

$$w(0) = -(Pr_0^2/16\pi D)f_1(\lambda) \quad (79)$$

$$M_{rr}(r_0) = (P/4\pi)f_2(\lambda) \quad (80)$$

where

$$f_1(\lambda) = (8/\lambda^2)\{\pi/4 + \mathrm{kei}\,\lambda + C_1(\mathrm{ber}\,\lambda - 1) + C_2\,\mathrm{bei}\,\lambda\} \quad (81)$$

$$f_2(\lambda) = 2\{-\ker\lambda + C_1\,\mathrm{bei}\,\lambda - C_2\,\mathrm{ber}\,\lambda\} \quad (82)$$

When $\lambda = 0$ the functions f_1 and f_2 must assume the values unity in order that the corresponding formulas of the theory of flat plates are obtained.

The following table contains the results of calculations which have been carried out.[10]

TABLE I

λ	$Wr(\lambda)$	$Ww(\lambda)$	$h=0$		$h=\infty, \nu=\frac{1}{3}$		$h=0$		$h=\infty, \nu=\frac{1}{3}$		$\frac{\lambda^2}{4\sqrt{3}} = \sqrt{1+\frac{h}{l}} - 1$
			c_1	c_2	c_1	c_2	$f_1(\lambda)$	$f_2(\lambda)$	$f_1(\lambda)$	$f_2(\lambda)$	
0							1.000	1.000	1.000	1.000	.00
0.1	.1459	−4.961	.7844	−2.918	.7837	−2.918	.999	.999	.999	.999	.00
0.2	.2225	−2.422	.7817	−2.225	.7782	−2.225	.999	.999	.999	.999	.01
0.5	.3276	−.811	.7626	−1.309	.7412	−1.310	.999	.999	.999	.999	.03
1.0	.3156	−.165	.6920	−.6215	.6111	−.6317	.996	.995	.985	.975	.14
2.0	.0654	.149	.4328	−.0070	.2160	−.1235	.935	.932	.817	.690	.58
3.0	−.1006	.0567	.1410	.1469	.0220	−.0652	.754	.746	.515	.191	1.30
4.0	−.0544	−.0691	−.0063	.0888	.0143	−.0424	.506	.498	.320	−.080	2.31
5.0	.0481	−.0512	−.0281	.0246	.0149	−.0133	.321	.324	.220	−.140	3.62
6.0	.0484	.0332	−.0159	−.0000	.0073	−.0007	.210	.234	.161	−.117	5.20
7.0	−.0220	.0454	−.0054	−.0049	.0020	.0014	.148	.192	.122	−.080	7.10
8.0	−.0422	.0125	−.0006	−.0031	.0002	.0008	.111	.168	.095	−.059	9.25
9.0	.0055	−.0388	.0007	−.0013	−.0001	.0003	.085	.153	.075	−.034	11.7
10.0	.0353	−.0004	.0007	−.0002	−.0001	.0000	.069	.140	.061	−.026	14.0

[10] These calculations have been carried out with two more decimals than are listed in the table. We have employed no other checks, except to assure ourselves of the correct limiting behavior of the functions f_1 and f_2 and of the smoothness of the curves representing f_1 and f_2.

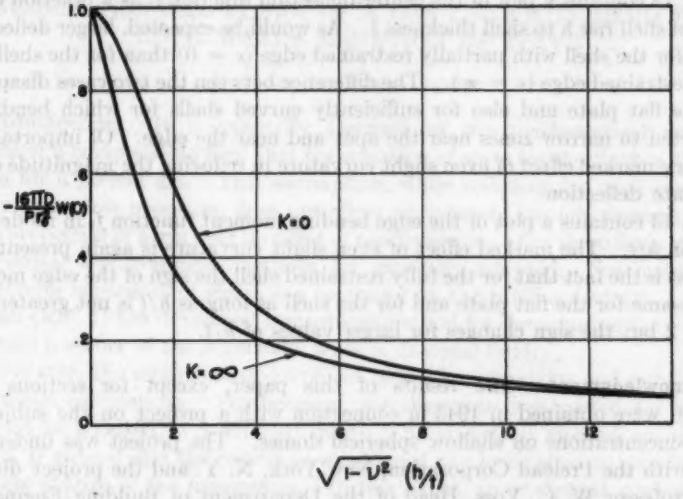

FIG. 12. Center deflection due to point load at center for shell with edge built-in ($\kappa = \infty$, $\nu = \frac{1}{4}$) and for shell with edge restrained against rotation but not against expansion ($\kappa = 0$).

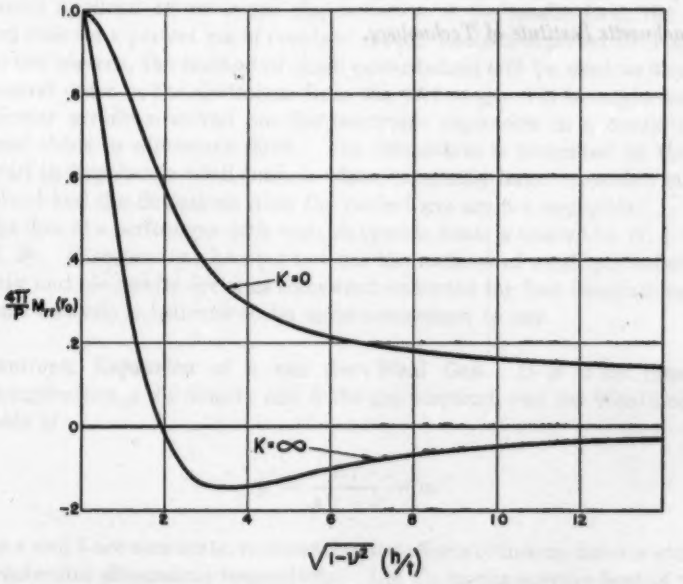

FIG. 13. Edge bending moment due to point load at center for shell with edge built in ($\kappa = \infty$, $\nu = \frac{1}{4}$) and for shell with edge restrained against rotation but not against expansion ($\kappa = 0$).

Fig. 12 contains a plot of the center-deflection function f_1 as a function of the ratio of shell rise h to shell thickness t. As would be expected, larger deflections occur for the shell with partially restrained edge ($\kappa = 0$) than for the shell with fully restrained edge ($\kappa = \infty$). The difference between the two cases disappears for the flat plate and also for sufficiently curved shells for which bending is restricted to narrow zones near the apex and near the edge. Of importance is the very marked effect of even slight curvature in reducing the magnitude of the flat-plate deflection.

Fig. 13 contains a plot of the edge bending moment function f_2 in its dependence on h/t. The marked effect of even slight curvature is again present. Of interest is the fact that for the fully restrained shell the sign of the edge moment is the same for the flat plate and for the shell as long as h/t is not greater than about 2 but the sign changes for larger values of h/t.

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ONE-DIMENSIONAL FLOWS OF A GAS CHARACTERIZED BY VAN DER WAAL'S EQUATION OF STATE

By Hsue-shen Tsien

Introduction. It is generally assumed in the mathematical analysis of flows of a compressible fluid that the equation of state for the fluid is the equation for a perfect gas. This assumption, while sufficient for most cases of air flow under low pressures, does not allow an accurate enough representation of the properties of a gas near its point of condensation. For these flows of vapors, numerical and graphical methods are generally adopted. For instance, the isentropic expansion of steam can be traced on the Mollier diagram. A. Busemann (Ref. 1) has developed a graphical method for determining the expansion around a corner of the supersonic flows of any real fluid.

However, the numerical or graphical methods are tedious and involve many laborious calculations. Furthermore, the diagrams constructed for one fluid cannot be adopted to a different fluid. The analytical method, on the other hand, is general and the results so established can be applied to any fluid with a mere change of the parameters. In this paper, an analytical method of calculating one-dimensional flows of a gas characterized by van der Waal's equation of state will be presented. The specific heat of the gas at constant volume is assumed to vary only with the temperature according to a quadratic law. Since for most applications envisaged the deviation of the results from the conventional ones for a perfect gas of constant specific heats is expected to be not more than ten percent, the method of small perturbation will be used so that terms of second order in the deviations from the perfect gas will be neglected. The particular problems solved are the isentropic expansion in a nozzle and the normal shock in supersonic flows. The calculation is prompted by the recent interest in hypersonic wind tunnels where extremely large expansion ratios are involved and the deviations from the perfect gas are not negligible.

The flow of a perfect gas with variable specific heats is treated by W. J. Walker (Ref. 2). It seems that he does not use the method of small perturbation explicitly and his results are thus somewhat awkward for flow calculations. The present analysis is believed to be more convenient to use.

Isentropic Expansion of a van der Waal Gas. If p is the pressure, T the temperature, ρ the density and R the gas constant, van der Waal's equation of state is

$$p = \frac{RT}{\rho^{-1} - b} - a\rho^2 \tag{1}$$

where a and b are constants, representing the effects of intermolecular attraction and molecular dimensions respectively. Let C_v be the specific heat of the gas at constant volume, assumed to be a function of temperature only. Then for

isentropic expansion, the first law of thermodynamics requires that

$$C_v dT + p d\rho^{-1} = C_v dT - p d\rho/\rho^2 = 0 \tag{2}$$

By substituting equation (1) into equation (2), one has

$$C_v dT - \{RT(\rho - b\rho^2)^{-1} - a\} d\rho = 0 \tag{3}$$

Generally, the variation of the specific heat C_v can be quite accurately represented by the following expression:

$$C_v = \alpha + \beta T + \gamma T^2 \tag{4}$$

where α, β, γ are constants. Then equation (3) can be written as

$$(\alpha + \beta T + \gamma T^2) dT - \{RT(\rho - b\rho^2)^{-1} - a\} d\rho = 0 \tag{5}$$

This equation is the differential law for the variation of T with ρ in an isentropic expansion.

To integrate equation (5), the calculations are greatly simplified by the assumption that quantities in second order of a, b, β, γ can be neglected. This is justified on the ground that a, b, β, γ are the deviation from a perfect gas and these deviations are not expected to exceed ten per cent for most applications. With this assumption, the temperature T as a function of the density ρ can be expressed in the following form:

$$T(\rho) = T^{(0)}(\rho) + aT^{(1)}(\rho) + bT^{(2)}(\rho) + \beta T^{(3)}(\rho) + \gamma T^{(4)}(\rho) \tag{6}$$

By substituting equation (6) into equation (5) and by equating the quantities independent of a, b, β, γ to zero, one has

$$\alpha dT^{(0)} - (R/\rho) T^{(0)} d\rho = 0 \tag{7}$$

This is the equation connecting the temperature and density for a perfect gas undergoing isentropic expansion. The equations for the correction terms $T^{(1)}$, $T^{(2)}$, $T^{(3)}$ and $T^{(4)}$ are obtained by substituting equation (6) into equation (5) and then equating separately the coefficients of a, b, β, and γ in the resulting expression to zero. Thus,

$$\alpha dT^{(1)} - \{(R/\rho) T^{(1)} - 1\} d\rho = 0 \tag{8}$$

$$\alpha dT^{(2)} - \{(R/\rho)(T^{(2)} + \rho T^{(0)})\} d\rho = 0 \tag{9}$$

$$\alpha dT^{(3)} + T^{(0)} dT^{(0)} - (R/\rho) T^{(3)} d\rho = 0 \tag{10}$$

$$\alpha dT^{(4)} + T^{(0)2} dT^{(0)} - (R/\rho) T^{(4)} d\rho = 0 \tag{11}$$

Equations (7) to (11) are now the linear first order differential equations for $T^{(0)}$, $T^{(1)}$, $T^{(2)}$, $T^{(3)}$ and $T^{(4)}$ respectively. Their solutions are

$$T^{(0)} = C\rho^{R/\alpha} \tag{12}$$

$$T^{(1)} = \rho/(R - \alpha) \tag{13}$$

$$T^{(2)} = (RC/\alpha)\rho^{(R/\alpha)+1} \tag{14}$$

$$T^{(3)} = -(C^2/\alpha)\rho^{2R/\alpha} \tag{15}$$

$$T^{(4)} = -(C^3/2\alpha)\rho^{3R/\alpha} \tag{16}$$

The constants of integrations for $T^{(1)}$, $T^{(2)}$, $T^{(3)}$ and $T^{(4)}$ are absorbed into the constant C. The temperature T according to equation (6) can then be written as

$$T = C\rho^{R/\alpha} + a\frac{\rho}{R-\alpha} + b\frac{CR}{\alpha}\rho^{R/\alpha+1} - \beta\frac{C^2}{\alpha}\rho^{2R/\alpha} - \gamma\frac{C^3}{2\alpha}\rho^{3R/\alpha} \tag{17}$$

To eliminate the constant C, the quantities connected with the starting point of the expansion can be utilized. Let these quantities be denoted by the subscript o. Then

$$T_0 = C\rho_o^{R/\alpha} + \alpha\frac{\rho_o}{R-\alpha} + b\frac{CR}{\alpha}\rho_o^{R/\alpha+1} - \beta\frac{C^2}{\alpha}\rho_o^{2R/\alpha} - \gamma\frac{C^3}{2\alpha}\rho_o^{3R/\alpha} \tag{18}$$

The constant C can be eliminated between equations (17) and (18). Then the temperature ratio T/T_0 can be written as

$$\frac{T}{T_o} = \left(\frac{\rho}{\rho_o}\right)^{\lambda}\left[1 + \left(\frac{a\rho_o}{RT_o}\right)\frac{\lambda}{1-\lambda}\left\{1 - \left(\frac{\rho}{\rho_o}\right)^{1-\lambda}\right\} - (b\rho_o)\lambda\left\{1 - \frac{\rho}{\rho_o}\right\}\right.$$
$$\left. + \left(\frac{\beta T_o}{\alpha}\right)\left\{1 - \left(\frac{\rho}{\rho_o}\right)^{\lambda}\right\} + \left(\frac{\lambda T_o^2}{\alpha}\right)\tfrac{1}{2}\left\{1 - \left(\frac{\rho}{\rho_o}\right)^{2\lambda}\right\}\right]$$

where λ is given by

$$\lambda = R/\alpha \tag{20}$$

If the specific heat C_V were constant, then $\beta = \gamma = 0$. If the gas were perfect, then $a = b = 0$. Then equation (19) would be reduced to the simple relation $T/T_o = (\rho/\rho_o)^{\lambda}$, the well-known equation for perfect gas.

By inverting equation (19) and retaining only terms of proper order of magnitudes, one has

$$\frac{\rho}{\rho_o} = \left(\frac{T}{T_o}\right)^{1/\lambda}\left[1 - \left(\frac{a\rho_o}{RT_o}\right)\frac{1}{1-\lambda}\left\{1 - \left(\frac{T}{T_o}\right)^{(1/\lambda)-1}\right\} + (b\rho_o)\left\{1 - \left(\frac{T}{T_o}\right)^{1/\lambda}\right\}\right.$$
$$\left. - \left(\frac{\beta T_o}{\alpha}\right)\frac{1}{\lambda}\left\{1 - \frac{T}{T_o}\right\} - \left(\frac{\gamma T_o^2}{\alpha}\right)\frac{1}{2\lambda}\left\{1 - \left(\frac{T}{T_o}\right)^2\right\}\right] \tag{21}$$

Other equations involving the pressure p can be easily obtained by using the equation of states (1). The results are

$$\frac{p}{p_o} = \left(\frac{\rho}{\rho_o}\right)^{\lambda+1}\left[1 + \left(\frac{a\rho_o}{RT_o}\right)\frac{1}{1-\lambda}\left\{1 - \left(\frac{\rho}{\rho_o}\right)^{1-\lambda}\right\} - (b\rho_o)(1+\lambda)\left\{1 - \left(\frac{\rho}{\rho_o}\right)\right\} + \left(\frac{\beta T_o}{\alpha}\right)\left\{1 - \left(\frac{\rho}{\rho_o}\right)^{\lambda}\right\} + \left(\frac{\gamma T_o^2}{\alpha}\right)\frac{1}{2}\left\{1 - \left(\frac{\rho}{\rho_o}\right)^{2\lambda}\right\}\right] \tag{22}$$

$$\frac{\rho}{\rho_o} = \left(\frac{p}{p_o}\right)^{1/(1+\lambda)}\left[1 - \left(\frac{a\rho_o}{RT_o}\right)\frac{1}{1-\lambda^2}\left\{1 - \left(\frac{p}{p_o}\right)^{(1-\lambda)/(1+\lambda)}\right\} + (b\rho_o)\left\{1 - \left(\frac{p}{p_o}\right)^{1/(1+\lambda)}\right\} - \left(\frac{\beta T_o}{\alpha}\right)\frac{1}{1+\lambda}\left\{1 - \left(\frac{p}{p_o}\right)^{\lambda/(1+\lambda)}\right\} - \left(\frac{\gamma T_o^2}{\alpha}\right)\frac{1}{2(1+\lambda)}\left\{1 - \left(\frac{p}{p_o}\right)^{2\lambda/(1+\lambda)}\right\}\right] \tag{23}$$

$$\frac{p}{p_o} = \left(\frac{T}{T_o}\right)^{1+(1/\lambda)}\left[1 - \left(\frac{a\rho_o}{RT_o}\right)\frac{\lambda}{1-\lambda}\left\{1 - \left(\frac{T}{T_o}\right)^{(1/\lambda)-1}\right\} - \left(\frac{\beta T_o}{\alpha}\right)\frac{1}{\lambda}\left\{1 - \frac{T}{T_o}\right\} - \left(\frac{\gamma T_o^2}{\alpha}\right)\frac{1}{2\lambda}\left\{1 - \left(\frac{T}{T_o}\right)^2\right\}\right] \tag{24}$$

$$\frac{T}{T_o} = \left(\frac{p}{p_o}\right)^{\lambda/(1+\lambda)}\left[1 + \left(\frac{a\rho_o}{RT_o}\right)\frac{\lambda^2}{1-\lambda^2}\left\{1 - \left(\frac{p}{p_o}\right)^{(1-\lambda)/(1+\lambda)}\right\} + \left(\frac{\beta T_o}{\alpha}\right)\frac{1}{1+\lambda}\left\{1 - \left(\frac{p}{p_o}\right)^{\lambda/(1+\lambda)}\right\} + \left(\frac{\gamma T_o^2}{\alpha}\right)\frac{1}{2(1+\lambda)}\left\{1 - \left(\frac{p}{p_o}\right)^{2\lambda/(1+\lambda)}\right\}\right] \tag{25}$$

Thus once the properties of the gas are known and the initial state of the gas specified, the "correction parameters" $(a\rho_o/RT_o)$, $(b\rho_o)$, $(\beta T_o/\alpha)$ and $(\gamma T_o^2/\alpha)$ can be immediately calculated. Then the subsequent states of expansion can be determined by equations (19) to (25). The functions of density ratios, pressure ratios and temperature ratios in these equations are universal for any gas and are independent of the initial conditions.

Expansion in a Nozzle. For isentropic expansion of the gas in a nozzle, one is interested in determining the velocity of flow. Let this velocity be denoted by v, then the dynamic equation of the flow is

$$\rho v dv = -dp \tag{26}$$

On the other hand, equation (2) can be rewritten as

$$C_v\, dT + p/\rho - dp/\rho = 0 \tag{27}$$

By combining equations (4), (26) and (27), the following differential equation is obtained:

$$(\alpha + \beta T + \gamma T^2)dT + d(p/\rho) + vdv = 0 \tag{28}$$

If the conditions at the starting point of the expansion, or the stagnation point, are denoted by the subscript o, the result of integrating equation (28) is

$$\tfrac{1}{2}v^2 = \alpha T_o\left\{1 - \frac{T}{T_o}\right\} + \frac{\beta T_o^2}{2}\left\{1 - \left(\frac{T}{T_o}\right)^2\right\} + \frac{\gamma T_o^3}{3}\left\{1 - \left(\frac{T}{T_o}\right)^3\right\} + \frac{p_o}{\rho_o}\left\{1 - \left(\frac{p}{p_o}\right)\left(\frac{\rho_o}{\rho}\right)\right\} \tag{29}$$

With the temperature ratio and pressure ratio determined by the equations given in the previous section, the velocity of the flow can be then calculated by equation (29).

For many applications such as wind tunnel design, the most convenient parameter for velocity is the Mach number M, the ratio of the velocity v and the velocity of sound c defined by

$$c^2 = dp/d\rho \tag{30}$$

where p and ρ are those in an isentropic process. This quantity can thus be calculated by differentiating equation (22). The result is

$$c^2 = (1+\lambda)\frac{p_o}{\rho_o}\left(\frac{\rho}{\rho_o}\right)^\lambda\left[1 + \left(\frac{a\rho_o}{RT_o}\right)\frac{1}{1-\lambda}\left\{1 - \frac{2}{1+\lambda}\left(\frac{\rho}{\rho_o}\right)^{1-\lambda}\right\} - (b\rho_o)(1+\lambda)\right. \tag{31}$$

$$\left.\left\{1 - \frac{2+\lambda}{1+\lambda}\left(\frac{\rho}{\rho_o}\right)\right\} + \left(\frac{\beta T_o}{\alpha}\right)\left\{1 - \frac{1+2\lambda}{1+\lambda}\left(\frac{\rho}{\rho_o}\right)^\lambda\right\} + \left(\frac{\lambda T_o^2}{\alpha}\right)\tfrac{1}{2}\left\{1 - \frac{1+3\lambda}{1+\lambda}\left(\frac{\rho}{\rho_o}\right)^{2\lambda}\right\}\right]$$

Expressed in terms of local conditions ρ, T, and ρ, the velocity of sound is given by

$$c^2 = (1+\lambda)\frac{p}{\rho}\left[1 - \left(\frac{a\rho}{RT}\right)\frac{1}{1+\lambda} + (b\rho) - \left(\frac{\beta T}{\alpha}\right)\frac{\lambda}{1+\lambda} - \left(\frac{\gamma T^2}{\alpha}\right)\frac{\lambda}{1+\lambda}\right] \tag{32}$$

Equation (32) can be obtained from equation (31) by the simple artifice of removing all the subscripts. This is the same as moving the reference point de- by the subscript o to the point concerned. Equation (32) is considerably different from the conventional formula for the velocity of sound,

$$c^2 = \left(1 + \frac{R}{C_v}\right)\frac{p}{\rho} \tag{33}$$

It can be, however, easily shown by means of equation (4) that equation (32) can be written as

$$c^2 = \left(1 + \frac{R}{C_v}\right)\frac{p}{\rho}\left[1 - \left(\frac{a\rho}{RT}\right)\frac{1}{1+\lambda} + (b\rho)\right]. \tag{34}$$

The difference between equations (33) and (34) is then purely due to the "thermal" imperfections of the gas.

In equation (29) the density ratio and the pressure ratio can be replaced as functions of the temperature ratio by equation (21) and (24). Then by replacing the velocity v by the Mach number M through equation (31), one obtains the relation between the temperature ratio T/T_0 and the Mach number:

$$\frac{T}{T_0} = \frac{1}{(1+\lambda M^2/2)}\left[1 + \left(\frac{a\rho_o}{RT_o}\right)t_1 - (b\rho_0)t_2 + \left(\frac{\beta T_o}{\alpha}\right)t_3 + \left(\frac{\gamma T_o^2}{\alpha}\right)t_4\right]. \tag{35}$$

Here t_1, t_2, t_3 and t_4 are functions of M and λ given by the following equations:

$$\left.\begin{aligned} t_1 &= \frac{\lambda}{\lambda+\lambda}\left\{M^2\left(1+\frac{\lambda}{2}M^2\right)^{-1/\lambda}+\left(1+\frac{\lambda}{2}M^2\right)^{1-(1/\lambda)}-1\right\} \\ t_2 &= \frac{1}{1+\lambda}\left\{(M^2-1)\left(1+\frac{\lambda}{2}M^2\right)^{-1-(1/\lambda)}+2\left(1+\frac{\lambda}{2}M^2\right)^{-1/\lambda}-1\right\} \\ t_3 &= \frac{1}{1+\lambda}\left\{\tfrac{1}{2}+\lambda\left(1+\frac{\lambda}{2}M^2\right)^{-1}-(\tfrac{1}{2}+\lambda)\left(1+\frac{\lambda}{2}M^2\right)^{-2}\right\} \\ t_4 &= \frac{1}{1+\lambda}\left\{\tfrac{1}{3}+\lambda\left(1+\frac{\lambda}{2}M^2\right)^{-2}-(\tfrac{1}{3}+\lambda)\left(1+\frac{\lambda}{2}M^2\right)^{-3}\right\}. \end{aligned}\right\} \quad (36)$$

It is easily recognized that the factor outside the bracket on the right side of equation (35) is the conventional value of the temperature ratio for a perfect gas. The functions t_1, t_2, t_3 and t_4 give then the corrections due to "thermal" and "calorical" imperfections.

By using the relations between the pressure ratio, the density ratio and the temperature ratio given in the previous section, the pressure ratio and the density ratio can be calculated in terms of the Mach number. These relations are the following:

$$\frac{p}{p_o}=\frac{1}{(1+\lambda M^2/2)^{1+(1/\lambda)}}\left[1+\left(\frac{a\rho_o}{RT_o}\right)P_1-(b\rho_o)P_2+\left(\frac{\beta T_o}{\alpha}\right)P_3+\left(\frac{\gamma T_o^2}{\alpha}\right)P_4\right] \quad (37)$$

$$\frac{\rho}{\rho_o}=\frac{1}{(1+\lambda M^2/2)^{1/\lambda}}\left[1+\left(\frac{a\rho_o}{RT_o}\right)r_1-(b\rho_o)r_2+\left(\frac{\beta T_o}{\alpha}\right)r_3+\left(\frac{\gamma T_o^2}{\alpha}\right)r_4\right] \quad (38)$$

where

$$\left.\begin{aligned} P_1 &= M^2\left(1+\frac{\lambda}{2}M^2\right)^{-1/\lambda}-\frac{1}{1-\lambda}\left\{1-\left(1+\frac{\lambda}{2}M^2\right)^{1-(1/\lambda)}\right\} \\ P_2 &= (M^2-1)\left(1+\frac{\lambda}{2}M^2\right)^{-1-(1/\lambda)}+2\left(1+\frac{\lambda}{2}M^2\right)^{1/\lambda}-1 \\ P_3 &= (\tfrac{1}{2}M^2-1)\left(1+\frac{\lambda}{2}M^2\right)^{-2}+\left(1+\frac{1}{2\lambda}\right)\left(1+\frac{\lambda}{2}M^2\right)^{-1}-\frac{1}{2\lambda} \\ P_4 &= (\tfrac{1}{3}M^2-1)\left(1+\frac{\lambda}{2}M^2\right)^{-3} \\ &\quad +\left(1-\frac{1}{3\lambda}\right)\left(1+\frac{\lambda}{2}M^2\right)^{-2}+\frac{1}{2\lambda}\left(1+\frac{\lambda}{2}M^2\right)^{-1}-\frac{1}{6\lambda} \end{aligned}\right\} \quad (39)$$

and

$$\left.\begin{aligned}
r_1 &= \frac{1}{1+\lambda} M^2 \left(1+\frac{\lambda}{2}M^2\right)^{-1/\lambda} - \frac{2}{1-\lambda^2}\left\{1-\left(1+\frac{\lambda}{2}M^2\right)^{1-(1/\lambda)}\right\} \\
r_2 &= \frac{1}{1+\lambda}(M^2-1)\left(1+\frac{\lambda}{2}M^2\right)^{-1-(1/\lambda)} \\
&\quad + \left(1+\frac{2}{1+\lambda}\right)\left(1+\frac{\lambda}{2}M^2\right)^{-1/\lambda} - \left(1+\frac{1}{1+\lambda}\right) \\
r_3 &= \frac{1}{1+\lambda}\left(\frac{M^2}{4}-1\right)\left(1+\frac{\lambda}{2}M^2\right)^{-2} \\
&\quad + \frac{4\lambda+1}{2\lambda(1+\lambda)}\left(1+\frac{\lambda}{2}M^2\right)^{-1} - \frac{2\lambda+1}{2\lambda(1+\lambda)} \\
r_4 &= \frac{1}{1+\lambda}\left(\frac{M^2}{6}-1\right)\left(1+\frac{\lambda}{2}M^2\right)^{-2} \\
&\quad + \frac{9\lambda+1}{6\lambda(1+\lambda)}\left(1+\frac{\lambda}{2}M^2\right)^{-2} - \frac{3\lambda+1}{6\lambda(1+\lambda)}.
\end{aligned}\right\} \tag{40}$$

Here again the functions P_1, P_2, P_3, P_4 and r_1, r_2, r_3, r_4 are correction functions for the pressure ratio and the density ratio respectively.

To calculate the cross-sectional area A of the nozzle, one needs the equation of continuity,

$$d(\rho v A) = 0 \tag{41}$$

From equation (41), the change of the area A can be expressed as follows

$$\frac{dA}{A} = -\left(\frac{d\rho}{\rho} + \frac{dv}{v}\right). \tag{42}$$

But by using equation (26), the first term on the right of above equation is equal to

$$\frac{d\rho}{\rho} = \frac{d\rho}{dp}\frac{dp}{\rho} = -\frac{v^2}{c^2}\frac{dv}{v} = -M^2\frac{dv}{v}.$$

Therefore, equation (42) can be written as

$$dA/A = -(1-M^2)dv/v \tag{43}$$

This equation shows that at the throat of the nozzle where $dA = 0$, the local Mach number is unity for any equation of state. This is the well-known result generally deduced for flow of a perfect gas.

Equation (41) requires that $\rho v A$ = constant. Therefore, if quantities at the throat are denoted by an asterisk, then

$$\frac{A}{A^*} = \frac{\rho^* v^*}{\rho v} = \left(\frac{\rho^*}{\rho_0}\right)\left(\frac{\rho_0}{\rho}\right)\left(\frac{v^*}{\sqrt{p_0/\rho_0}}\right)\left(\frac{\sqrt{p_0/\rho_0}}{c}\right)\frac{1}{M}. \tag{44}$$

All quantities entering into this equation can be expressed in terms of the Mach number M by using the formulae previously derived. The result of this calculation, retaining only first order terms of a, b, β, and γ, is

$$\frac{A}{A^*} = \frac{1}{M}\left(\frac{1+\lambda M^2/2}{1+(\lambda/2)}\right)^{1+(1/\lambda)} \cdot \left[1 + \left(\frac{a\rho_o}{RT_o}\right) f_1 - (b\rho_o) f_2 + \left(\frac{\beta T_o}{\alpha}\right) f_3 + \left(\frac{\gamma T_o^2}{\alpha}\right) f_4\right] \tag{45}$$

where

$$f_1 = \frac{1+(\lambda/2)}{1+\lambda}\left[\left(1+\frac{\lambda}{2}\right)^{-1/\lambda} - M^2\left(1+\frac{\lambda}{2}M^2\right)^{-1/\lambda}\right] + \frac{1}{1-\lambda}\left[\left(1+\frac{\lambda}{2}\right)^{1-(1/\lambda)} - \left(1+\frac{\lambda}{2}M^2\right)^{1-(1/\lambda)}\right]$$

$$f_2 = \frac{1+(\lambda/2)}{1+\lambda}\left[(2+\lambda)\left(1+\frac{\lambda}{2}\right)^{-1-(1/\lambda)} - \{1+(1+\lambda)M^2\}\left(1+\frac{\lambda}{2}M^2\right)^{-1-(1/\lambda)}\right]$$

$$f_3 = \frac{(1+(\lambda/2))(\frac{1}{4}+(\lambda/2))}{(1+\lambda)}\left[\left(1+\frac{\lambda}{2}\right)^{-2} - M^2\left(1+\frac{\lambda}{2}M^2\right)^{-2}\right] + \frac{1}{4}\frac{(1+(\lambda/2))}{(1+\lambda)}\left[\left(1+\frac{\lambda}{2}\right)^{-1} - M^2\left(1+\frac{\lambda}{2}M^2\right)^{-1}\right] + \frac{1+\lambda-(\lambda^2/2)}{\lambda(1+\lambda)}\left[\left(1+\frac{\lambda}{2}\right)^{-1} - \left(1+\frac{\lambda}{2}M^2\right)^{-1}\right] \tag{46}$$

$$f_4 = \frac{(1+(\lambda/2))(\frac{1}{6}+(\lambda/2))}{(1+\lambda)}\left[\left(1+\frac{\lambda}{2}\right)^{-3} - M^2\left(1+\frac{\lambda}{2}M^2\right)^{-3}\right] + \frac{1}{6}\frac{(1+(\lambda/2))}{(1+\lambda)}\left[\left(1+\frac{\lambda}{2}\right)^{-2} - M^2\left(1+\frac{\lambda}{2}M^2\right)^{-2}\right] + \frac{1+\lambda-\lambda^2}{2\lambda(1+\lambda)}\left[\left(1+\frac{\lambda}{2}\right)^{-2} - \left(1+\frac{\lambda}{2}M^2\right)^{-2}\right] + \frac{1}{6}\frac{(1+(\lambda/2))}{(1+\lambda)}\left[\left(1+\frac{\lambda}{2}\right)^{-1} - M^2\left(1+\frac{\lambda}{2}M^2\right)^{-1}\right].$$

All characteristics of nozzle flow are thus given by equations (35), (37), (38) and (45). In these equations, the corrections introduced by the imperfections of the gas are separated into a series of terms, each of which is a product of two factors. One factor is a universal function of the Mach number M and the ratio $\lambda = R/\alpha$, but independent of the initial conditions of the expansion. The other factor is determined by these initial conditions only. This is a considerable simplification allowed by the small perturbation method adopted.

Shock Wave in Supersonic Flow. In addition to flow through a convergent-divergent nozzle, one dimensional flows of a gas have the important feature

of the formation of normal shock waves in the supersonic region. If the Mach number before the shock is very large, the pressure ratio across the shock is also very large. Then one must expect considerable deviations from the imperfections of the gas. To determine these deviations, consider the shock as fixed in space and denote the conditions ahead of the shock by the subscript 1 and conditions after the shock by the subscript 2. The flow velocities ahead and after the shock, being normal to the shock front, are thus v_1 and v_2 respectively. The equation of continuity is then

$$\rho_1 v_1 = \rho_2 v_2 \tag{47}$$

The equation of momentum is

$$\rho_1 v_1^2 - \rho_2 v_2^2 = p_2 - p_1 \tag{48}$$

The equation for the conservation of energy is, according to equation (28),

$$\alpha T_1 + \frac{\beta}{2} T_1^2 + \frac{\gamma}{3} T_1^3 + \frac{p_1}{\rho_1} + \tfrac{1}{2} v_1^2 = \alpha T_2 + \frac{\beta}{2} T_2^2 + \frac{\gamma}{3} T_2^3 + \frac{p_2}{\rho_2} + \tfrac{1}{2} v_2^2. \tag{49}$$

The equation of states is, of course, the van der Waal equation

$$p_2 = \frac{RT_2\rho_2}{1 - b\rho_2} - a\rho_2^2. \tag{50}$$

Equations (47) to (50) form the system of four simultaneous equations for the four unknowns v_2, ρ_2, p_2, T_2.

For flow calculations, it is generally more convenient to express the velocities as Mach numbers. With the aid of equation (32), equations (47), (48) and (49) can be rewritten as the following:

$$\begin{aligned}\rho_1 M_1^2(1+\lambda)p_1\left[1 - \frac{1}{1+\lambda}\frac{a\rho_1}{RT_1} + b\rho_1 - \frac{\lambda}{1+\lambda}\frac{\beta T_1}{\alpha} - \frac{\lambda}{1+\lambda}\frac{\gamma T_1^2}{\alpha}\right] \\ = \rho_2 M_2^2(1+\lambda)p_2\left[1 - \frac{1}{1+\lambda}\frac{a\rho_2}{RT_2} + b\rho_2 - \frac{\lambda}{1+\lambda}\frac{\beta T_2}{\alpha} - \frac{\lambda}{1+\lambda}\frac{\gamma T_2^2}{\alpha}\right]\end{aligned} \tag{51}$$

$$\begin{aligned}M_1^2(1+\lambda)p_1\left[1 - \frac{1}{1+\lambda}\frac{a\rho_1}{RT_1} + b\rho_1 - \frac{\lambda}{1+\lambda}\frac{\beta T_1}{\alpha} - \frac{\lambda}{1+\lambda}\frac{\gamma T_1^2}{\alpha}\right] \\ - M_2^2(1+\lambda)p_2\left[1 - \frac{1}{1+\lambda}\frac{a\rho_2}{RT_2} + b\rho_2 - \frac{\lambda}{1+\lambda}\frac{\beta T_2}{\alpha} - \frac{\lambda}{1+\lambda}\frac{\gamma T_2^2}{\alpha}\right] = p_2 - p_1\end{aligned} \tag{52}$$

$$\begin{aligned}\alpha T_1 + \frac{\beta}{2} T_1^2 + \frac{\gamma}{3} T_1^3 + \frac{p_1}{\rho_1} + \tfrac{1}{2} M_1^2(1+\lambda)\frac{p_1}{\rho_1} \\ \cdot\left[1 - \frac{1}{1+\lambda}\frac{a\rho_1}{RT_1} + b\rho_1 - \frac{\lambda}{1+\lambda}\frac{\beta T_1}{\alpha} - \frac{\lambda}{1+\lambda}\frac{\gamma T_1^2}{\alpha}\right] \\ = \alpha T_2 + \frac{\beta}{2} T_2^2 + \frac{\gamma}{3} T_2^3 + \frac{p_2}{\rho_2} + \tfrac{1}{2} M_2^2(1+\lambda)\frac{p_2}{\rho_2} \\ \cdot\left[1 - \frac{1}{1+\lambda}\frac{a\rho_2}{RT_2} + b\rho_2 - \frac{\lambda}{1+\lambda}\frac{\beta T^2}{\alpha} - \frac{\lambda}{1+\lambda}\frac{\gamma T_2^2}{\alpha}\right].\end{aligned} \tag{53}$$

In the system of equations (50) to (53), one can, for instance, eliminate the variables ρ_2, T_2 and M_2 and obtain an equation for p_2. This process of elimination is somewhat tedious, but nevertheless is straightforward. The result, retaining only terms linear in a, b, β and γ, is

$$\begin{aligned}
&\left[\frac{2+\lambda}{2}\left(\frac{p_2}{p_1}\right)^2-\{(1+\lambda)M_1^2+1\}\left(\frac{p_2}{p_1}\right)+\left\{(1+\lambda)M_1^2-\frac{\lambda}{2}\right\}\right]\\
&\quad+\left(\frac{a\rho_1}{RT_1}\right)\left[(1+\lambda)M_1^2\left\{1-\frac{1}{1-\dfrac{1}{(1+\lambda)M_1^2}\left(\dfrac{p_2}{p_1}-1\right)}\right\}+M_1^2\left(\frac{p_2}{p_1}-1\right)\right.\\
&\quad\left.+\frac{\dfrac{2+\lambda}{2}\left(\dfrac{p_2}{p_1}\right)^2-\{(1+\lambda)M_1^2+1\}\left(\dfrac{p_2}{p_1}\right)+\left\{(1+\lambda)M_1^2-\dfrac{\lambda}{2}\right\}}{1+\lambda}\right]\\
&\quad-(b\rho_1)\left[\frac{2+\lambda}{2}\left(\frac{p_2}{p_1}\right)^2-\{1+(1+\lambda)M_1^2\}\left(\frac{p_2}{p_1}\right)+\left\{(1+\lambda)M_1^2-\frac{\lambda}{2}\right\}\right]\\
&\quad+\left(\frac{\beta T_1}{\alpha}\right)\left[(1+\lambda)M_1^2\left\{\tfrac{1}{2}-\tfrac{1}{2}\left(\frac{p_2}{p_1}\right)^2\left[1-\frac{\dfrac{p_2}{p_1}-1}{(1+\lambda)M_1^2}\right]^2+\frac{\lambda}{1+\lambda}\left(\frac{p_2}{p_1}-1\right)\right\}\right.\\
&\quad\left.+\frac{\dfrac{2+\lambda}{2}\left(\dfrac{p_2}{p_1}\right)^2-\{(1+\lambda)M_1^2+1\}\left(\dfrac{p_2}{p_1}\right)+\left\{(1+\lambda)M_1^2-\dfrac{\lambda}{2}\right\}}{(1+\lambda)/\lambda}\right]\\
&\quad+\left(\frac{\gamma T_1^2}{\alpha}\right)\left[(1+\lambda)M_1^2\left\{\tfrac{1}{3}-\tfrac{1}{3}\left(\frac{p_2}{p_1}\right)^3\left[1-\frac{\dfrac{p_2}{p_1}-1}{(1+\lambda)M_1^2}\right]^3+\frac{\lambda}{1+\lambda}\left(\frac{p_2}{p_1}-1\right)\right\}\right.\\
&\quad\left.+\frac{\dfrac{2+\lambda}{2}\left(\dfrac{p_2}{p_1}\right)^2-\{(1+\lambda)M_1^2+1\}\left(\dfrac{p_2}{p_1}\right)+\left\{(1+\lambda)M_1^2-\dfrac{\lambda}{2}\right\}}{(1+\lambda)\lambda}\right]=0
\end{aligned}\tag{54}$$

According to the principle of small perturbation method adopted here, the pressure ratio (p_2/p_1) can be expressed as

$$\begin{aligned}
\left(\frac{p_2}{p_1}\right)=\left(\frac{p_2}{p_1}\right)^{(0)}&+\left(\frac{a\rho_1}{RT_1}\right)\left(\frac{p_2}{p_1}\right)^{(1)}\\
&-(b\rho_1)\left(\frac{p_2}{p_1}\right)^{(2)}+\left(\frac{\beta T_1}{\alpha}\right)\left(\frac{p_2}{p_1}\right)^{(3)}+\left(\frac{\gamma T_1^2}{\alpha}\right)\left(\frac{p_2}{p_1}\right)^{(4)}
\end{aligned}\tag{55}$$

where none of the functions $(p_2/p_1)^{(i)}(i = 0, 1, 2, 3, 4)$ contains the parameter a, b, β or γ. By substituting equation (55) into equation (54) and by equating the terms independent of a, b, β or γ to zero, one has the equation for the first approximation $(p_2/p_1)^{(0)}$ as

$$\frac{2+\lambda}{2}\left(\frac{p_2}{p_1}\right)^{(0)2}-\{(1+\lambda)M_1^2+1\}\left(\frac{p_2}{p_1}\right)^{(0)}+\left\{(1+\lambda)M_1^2-\frac{\lambda}{2}\right\}=0 \tag{56}$$

The significant root of this quadratic equation is

$$\left(\frac{p_2}{p_1}\right)^{(0)} = \frac{1}{2+\lambda}\,[2(1+\lambda)M_1^2 - \lambda] \tag{57}$$

This is the well-known result for shock wave of a perfect gas. The correction for intermolecular attraction can be determined by equating to zero the terms with the factor $(a\rho_1/RT_1)$ in the result of substituting (55) into equation (54). Thus

$$\left[(2+\lambda)\left(\frac{p_2}{p_1}\right)^{(0)} - \{(1+\lambda)M_1^2+1\}\right]\left(\frac{p_2}{p_1}\right)^{(1)} + (1+\lambda)M_1^2 \cdot\left\{1 - \frac{1}{1-\dfrac{1}{(1+\lambda)M_1^2}\left[\left(\frac{p_2}{p_1}\right)^{(0)}-1\right]}\right\} + M_1^2\left[\left(\frac{p_2}{p_1}\right)^{(0)}-1\right] = 0 \tag{58}$$

Similarly, one has

$$\left[(2+\lambda)\left(\frac{p_2}{p_1}\right)^{(0)} - \{(1+\lambda)M_1^2+1\}\right]\left(\frac{p_2}{p_1}\right)^{(2)} = 0 \tag{59}$$

$$\left[(2+\lambda)\left(\frac{p_2}{p_1}\right)^{(0)} - \{(1+\lambda)M_1^2+1\}\right]\left(\frac{p_2}{p_1}\right)^{(3)} + (1+\lambda)M_1^2 \cdot\left\{\tfrac{1}{2} - \tfrac{1}{2}\left(\frac{p_2}{p_1}\right)^{(0)2}\left[1 - \frac{\left(\frac{p_2}{p_1}\right)^{(0)}-1}{(1+\lambda)M_1^2}\right]^2 + \frac{\lambda}{1+\lambda}\left[\left(\frac{p_2}{p_1}\right)^{(0)}\right]\right\} = 0 \tag{60}$$

$$\left[(2+\lambda)\left(\frac{p_2}{p_1}\right)^{(0)} - \{(1+\lambda)M_1^2+1\}\right]\left(\frac{p_2}{p_1}\right)^{(4)} + (1+\lambda)M_1^2 \cdot\left\{\tfrac{1}{2} - \tfrac{1}{2}\left(\frac{p_2}{p_1}\right)^{(0)2}\left[1 - \frac{\left(\frac{p_2}{p_1}\right)^{(0)}-1}{(1+\lambda)M_1^2}\right]^2 + \frac{\lambda}{1+\lambda}\left[\left(\frac{p_2}{p_1}\right)^{(0)}-1\right]\right\} = 0 \tag{61}$$

Since the value of $(p_2/p_1)^{(0)}$ is given by equation (57), these equations are linear equations for the functions $(p_2/p_1)^{(i)}(i = 1, 2, 3, 4)$. They can be easily solved. The final result for (p_2/p_1) can be written in the following form:

$$\frac{p_2}{p_1} = \frac{2(1+\lambda)M_1^2 - \lambda}{(2+\lambda)} \cdot\left[1 + \left(\frac{a\rho_1}{RT_1}\right)\Pi_1 - (b\rho_1)\Pi_2 + \left(\frac{\beta T_1}{\alpha}\right)\Pi_3 + \left(\frac{\gamma T_1^2}{\alpha}\right)\Pi_4\right] \tag{62}$$

where

$$\left.\begin{aligned}
\Pi_1 &= -\frac{2\lambda M_1^2(M_1^2-1)}{(\lambda M_1^2+2)[2(1+\lambda)M_1^2-\lambda]}, \qquad \Pi_2 = 0 \\
\Pi_3 &= \frac{2\lambda M_1^2}{[2(1+\lambda)M_1^2-\lambda]}\left[\frac{1}{(2+\lambda)^2}\left\{(1+\lambda)+\frac{1}{M_1^2}\right\}\right. \\
&\qquad \left.\cdot\left\{\lambda(1+\lambda)M_1^2+4(1+\lambda)-\frac{\lambda}{M_1^2}\right\}-1\right] \\
\Pi_4 &= \frac{2\lambda M_1^2}{[2(1+\lambda)M_1^2-\lambda]}\left[\frac{1}{(2+\lambda)^3}\frac{1}{2}\left\{(1+\lambda)+\frac{1}{M_1^2}\right\}\left\{4\lambda^2(1+\lambda)^2M_1^4\right.\right. \\
&\quad + 2\lambda(1+\lambda)(12+12\lambda-\lambda^2)M_1^2+[(4+4\lambda-\lambda^2)^2+32(1+\lambda)^2] \\
&\qquad \left.\left.-2\lambda(12+12\lambda-\lambda^2)\frac{1}{M_1^2}+\frac{4\lambda^2}{M_1^4}\right\}-1\right]
\end{aligned}\right\} \tag{63}$$

By using equation (62) and previously given relations (50) to (53), the density ratio ρ_2/ρ_1, the temperature ratio T_2/T_1 and the Mach number M_2 can be determined. This calculation gives the following relations:

$$\frac{\rho_2}{\rho_1} = \frac{(2+\lambda)M_1^2}{\lambda M_1^2+2}\left[1+\left(\frac{a\rho_1}{RT_1}\right)D_1-(b\rho_1)D_2+\left(\frac{\beta T_1}{\alpha}\right)D_3+\left(\frac{\gamma T_1^2}{\alpha}\right)D_4\right] \tag{64}$$

where

$$\left.\begin{aligned}
D_1 &= \frac{2(1+\lambda)M_1^2-\lambda}{(1+\lambda)(\lambda M_1^2+2)}\Pi_1+\frac{2(M_1^2-1)}{(1+\lambda)(\lambda M_1^2+2)} \\
D_2 &= \frac{2(M_1^2-1)}{(\lambda M_1^2+2)} \\
D_3 &= \frac{2(1+\lambda)M_1^2-\lambda}{(1+\lambda)(\lambda M_1^2+2)}\Pi_3+\frac{2\lambda(M_1^2-1)}{(1+\lambda)(\lambda M_1^2+2)} \\
D_4 &= \frac{2(1+\lambda)M_1^2-\lambda}{(1+\lambda)(\lambda M_1^2+2)}\Pi_4+\frac{2\lambda(M_1^2-1)}{(1+\lambda)(\lambda M_1^2+2)}
\end{aligned}\right\} \tag{65}$$

$$\frac{T_2}{T_1} = \frac{\{2(1+\lambda)M_1^2-\lambda\}(\lambda M_1^2+2)}{(2+\lambda)^2M_1^2} \cdot\left[1+\left(\frac{a\rho_1}{RT_1}\right)\tau_1-(b\rho_1)\tau_2+\left(\frac{\beta T_1}{\alpha}\right)\tau_3+\left(\frac{\gamma T_1^2}{\alpha}\right)\tau_4\right] \tag{66}$$

where

$$\left.\begin{aligned}
\tau_1 &= \Pi_1-D_1+\frac{(2+\lambda)^2M_1^4}{(\lambda M_1^2+2)^2}\frac{(2+\lambda)}{\{2(1+\lambda)M_1^2-\lambda\}}-1 \\
\tau_2 &= 0, \qquad \tau_3 = \Pi_3-D_3, \qquad \tau_4 = \Pi_4-D_4
\end{aligned}\right\} \tag{67}$$

$$M_2 = \sqrt{\frac{\lambda M_1^2 + 2}{2(1+\lambda)M_1^2 - \lambda}} \cdot \left[1 + \left(\frac{a\rho_1}{RT_1}\right) m_1 - (b\rho_1) m_2 + \left(\frac{\beta T_1}{\alpha}\right) m_3 + \left(\frac{\gamma T_1^2}{\alpha}\right) m_4\right] \tag{68}$$

where

$$\left.\begin{aligned}
m_1 &= \frac{\left(1+\frac{\lambda}{2}\right)M_1^2}{(1+\lambda)(\lambda M_1^2+2)}\left[\frac{(2+\lambda)^3 M_1^4}{(\lambda M_1^2+2)^2\{2(1+\lambda)M_1^2-\lambda\}} - 1\right] \\
&\qquad - \frac{\left(1+\frac{\lambda}{2}\right)\left(M_1^2+\frac{1}{1+\lambda}\right)}{(\lambda M_1^2+2)}\,\Pi_1 \\
m_2 &= \frac{\left(1+\frac{\lambda}{2}\right)M_1^2}{(\lambda M_1^2+2)}\left[\frac{(2+\lambda)M_1^2}{(\lambda M_1^2+2)} - 1\right] \\
m_3 &= \frac{\lambda\left(1+\frac{\lambda}{2}\right)M_1^2}{(1+\lambda)(\lambda M_1^2+2)}\left[\frac{(\lambda M_1^2+2)\{2(1+\lambda)M_1^2-\lambda\}}{(2+\lambda)^2 M_1^2} - 1\right] \\
&\qquad - \frac{\left(1+\frac{\lambda}{2}\right)\left(M_1^2+\frac{1}{1+\lambda}\right)}{(\lambda M_1^2+2)}\,\Pi_3 \\
m_4 &= \frac{\lambda\left(1+\frac{\lambda}{2}\right)M_1^2}{(1+\lambda)(\lambda M_1^2+2)}\left[\frac{(\lambda M_1^2+2)^2\{2(1+\lambda)M_1^2-\lambda\}^2}{(2+\lambda)^4 M_1^4} - 1\right] \\
&\qquad - \frac{\left(1+\frac{\lambda}{2}\right)\left(M_1^2+\frac{1}{1+\lambda}\right)}{(\lambda M_1^2+2)}\,\Pi_4
\end{aligned}\right\} \tag{69}$$

Equations (62), (64), (66) and (68) show that all pertinent quantities for the shock can be calculated by applying a series of correction terms to the formulae for perfect gas. Similar to results obtained in the preceeding section, these correction terms are products of two factors. One factor is a universal function of the Mach number M and the ratio $\lambda = R/\alpha$, but independent of the initial conditions of the shock. The other factor is determined by these initial conditions only.

Parameters for Constants a and b. In the analysis of the previous sections, the two parameters $(a\rho)/RT$ and $(b\rho)$ appear in all the formulae for

computation. The constants a and b are connected with the critical pressure p_c and the critical temperature T_c in the following way: (Ref. 3)

$$p_c = \frac{1}{27}\frac{a}{b^2}, \qquad T_c = \frac{8}{27}\frac{a}{Rb} \tag{70}$$

By solving these equations for a and b, one has

$$a = \frac{27}{64}\frac{(RT_c)^2}{p_c}, \qquad b = \frac{1}{8}\frac{RT_c}{p_c} \tag{71}$$

Therefore, if only first order quantities in a, b are considered, then equation (71) and the equation of states give

$$\frac{a\rho}{RT} = \frac{27}{64}\left(\frac{T_c}{T}\right)^2\left(\frac{p}{p_c}\right) \tag{72}$$

and

$$b\rho = \frac{1}{8}\left(\frac{p}{p_c}\right)\left(\frac{T_c}{T}\right) \tag{73}$$

These forms of the parameters are believed to be more convenient in actual computations and they clearly bring out the non-dimensional character of the parameters.

Properties of Air. The experimentally determined values of a and b for air are given by J. Jeans (Ref. 4). From these constants, the "critical pressure" and the "critical temperature" can be computed by using equation (70). The values in engineering units are

$$p_c = 324.6 \text{ psia}, \qquad T_c = 181.1°R \tag{74}$$

Since air is a mixture of several gases, such critical pressure and critical temperature are not real quantities, but convenient substitutes for a and b in the van der Waal equation of state.

The specific heat of air at constant volume is given by J. H. Keenan and J. Kaye (Ref. 5) as a function of temperature. By fitting the tabulated data from $T = 300°F$ to $T = 1000°R$ to equation (4) with the method of least square, the constants α, β and γ are determined as

$$\begin{aligned} \alpha &= \quad 0.17457 \text{ B.t.u. per lb. per °F} \\ \beta &= -0.01877 \times 10^{-3} \text{ B.t.u. per lb. per (°F)}^2 \\ \gamma &= \quad 0.02441 \times 10^{-6} \text{ B.t.u. per lb. per (°F)}^3 \end{aligned} \tag{75}$$

Since the gas constant R for air is 0.0685 B.t.u. per lb. per °F,

$$\lambda = R/\alpha = 0.3920 \tag{76}$$

Tabulation of Functions for Air. To facilitate the application of results obtained in the previous sections, the universal correction functions specified by equations (36), (39), (40), (46), (63), (65), (67) and (69) are tabulated for air in Tables 1 to 8. The values of factors outside the bracket of equations

(35), (37), (38), (45), (62), (64), (66) and (68), which are the values of the different quantities for a perfect gas, are also given in the first columns of these tables. It is seen that the correction functions have appreciable magnitudes even for moderate Mach numbers. Thus even for a Mach number as low as 3, the deviations from the perfect gas cannot be neglected if the correction parameters are not extremely small.

Application of Results to a Hypersonic Wind Tunnel. As an example, the results of analysis will be applied presently to a hypersonic wind tunnel of Mach number 15 in its test section. Let the pressure at inlet to the nozzle be 600 psia, and the temperature 800°R. Then the values of correction parameters are

$$\frac{a\rho_o}{RT_o} = \frac{27}{64}\left(\frac{181.1}{800}\right)^2 \frac{600}{324.6} = 0.0396$$

$$b\rho_o = \frac{1}{8}\frac{181.1}{800}\frac{600}{324.6} = 0.05230$$

$$\frac{\beta T_o}{\alpha} = -\frac{0.01877 \times 0.800}{0.17457} = -0.08602$$

$$\frac{\gamma T_o^2}{\alpha} = \frac{0.02441 \times 0.800^2}{0.17457} = 0.08949$$

With the aid of Tables 1 to 4, the conditions of air at the test section can be calculated as follows:

$$T = 800 \times 0.0221729\,(1 - 0.03996 \times 0.27702 + 0.05230 \times 0.28149 - 0.08602 \times 0.36512 + 0.08949 \times 0.23960) = 17.626°R$$

$$p = 600 \times 0.00000133655(1 - 0.03996 \times 1.62670 + 0.05230 \times 0.99958 + 0.08602 \times 1.19789 - 0.08949 \times 0.39642) = 0.00084591 \text{ psia}$$

$$A/A^* = 6791.55\,(1 + .03996 \times 1.77417 - 0.05230 \times 1.08814 - 0.08602 \times 1.29528 + 0.08949 \times 0.46368) = 6411.6$$

Considerable deviations from the values for a perfect gas are thus present.

Now if there is a normal shock wave immediately after the test section, Tables 5 to 8 can be used to calculate conditions after the shock wave. Here the correction parameters are

$$\frac{a\rho_1}{RT_1} = \frac{27}{64}\left(\frac{181.1}{17.626}\right)^2 \frac{0.00084591}{324.6} = 0.00012$$

$$b\rho_1 = \frac{1}{8}\frac{181.1}{17.626}\frac{0.00084591}{324.6} = 0.0000034$$

$$\frac{\beta T_1}{\alpha} = -\frac{0.01877 \times 0.017626}{0.17456} = -0.0018952$$

$$\frac{\gamma T_1^2}{\alpha} = \frac{0.02441 \times 0.017626^2}{0.17457} = 0.00004344$$

Then the temperature and the pressure can be calculated as follows:

$$T_2 = 17.626 \times 43.8613\ (1 - 0.00012 \times 1.64286 + 0.0018952 \times 14.983 - 0.00004344 \times 430.55) = 780.44°R$$

$$p_2 = 0.00084591 \times 261.709\ (1 - 0.00012 \times 0.69978 - 0.0018952 \times 3.4082 + 0.00004344 \times 107.67) = 0.22097 \text{ psia}$$

$$M_2 = 0.379589\ (1 + 0.00012 \times 0.24269 - 0.0000034 \times 14.818 - 0.0018952 \times 25.809 + 0.00004344 \times 1293.2) = 0.38234$$

The deviations from the values for a perfect gas are much smaller here than for the expansion. The reason is, of course, the much lower pressure level for the shock process. If a system of compressors is used to recompress the air after the shock to the pressure p_o at inlet to the nozzle so that the wind tunnel can be operated continuously, the overall compression ratio required is thus p_o/p_2, or approximately 2700.

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TABLE I

Functions for Temperature Ratio in Expansion of Air, cf. Equations (35) and (36)

M	$(1+\frac{1}{5}\lambda M^2)^{-1}$	t_1	t_2	t_3	t_4
0	1.00000	0	0	0	0
0.5	0.953289	0.04218	0.03870	0.04531	0.04397
1.0	0.836120	0.11012	0.07516	0.14667	0.13175
1.5	0.693963	0.12769	0.03636	0.24602	0.20094
2.0	0.560538	0.09041	−0.04482	0.31571	0.23617
2.5	0.449438	0.02866	−0.12201	0.35632	0.24904
3.0	0.361795	−0.03396	−0.17858	0.37720	0.25165
3.5	0.294031	−0.08750	−0.21578	0.38660	0.25056
4.0	0.241780	−0.13001	−0.23925	0.38982	0.24856
4.5	0.201248	−0.16271	−0.25392	0.38992	0.24662
5.0	0.169492	−0.18761	−0.26315	0.38852	0.24502
5.5	0.144321	−0.20656	−0.26905	0.38649	0.24376
6.0	0.124131	−0.22106	−0.27289	0.38428	0.24281
6.5	0.107747	−0.23225	−0.27544	0.38210	0.24208
7.0	0.0943040	−0.24097	−0.27716	0.38005	0.24153
7.5	0.0831601	−0.24784	−0.27835	0.37818	0.24111
8.0	0.0738334	−0.25329	−0.27918	0.37649	0.24079
8.5	0.0659587	−0.25767	−0.27977	0.37498	0.24054
9.0	0.0592557	−0.26121	−0.28020	0.37363	0.24034
9.5	0.0535074	−0.26411	−0.28052	0.37243	0.24019
10	0.0485437	−0.26650	−0.28076	0.37136	0.24007
11	0.0404596	−0.27014	−0.28107	0.36954	0.23989
12	0.0342185	−0.27271	−0.28126	0.36808	0.23977
13	0.0293049	−0.27459	−0.28137	0.36690	0.23969
14	0.0253704	−0.27597	−0.28144	0.36593	0.23964
15	0.0221729	−0.27702	−0.28149	0.36512	0.23960
16	0.0195404	−0.27783	−0.38152	0.36445	0.23957
17	0.0173479	−0.27846	−0.28155	0.36389	0.23955
18	0.0155029	−0.27896	−0.28156	0.36341	0.23953
19	0.0139361	−0.27936	−0.28157	0.36300	0.23952
20	0.0125956	−0.27969	−0.28158	0.36264	0.23951
∞	0	−0.28161	−0.28161	0.35920	0.23946

TABLE II

Functions for Pressure Ratio in Expansion of Air, cf. Equations (37) and (39)

M	$(1+\frac{1}{2}\lambda M^2)^{-1-(1/\lambda)}$	P_1	P_2	P_3	P_4
0	1.000000	0	0	0	0
0.5	0.843775	0.10367	0.13741	0.04175	0.09655
1.0	0.529633	0.23475	0.26688	0.10277	0.25853
1.5	0.273263	0.17452	0.12912	0.09292	0.32318
2.0	0.128024	−0.06100	−0.15914	0	0.27812
2.5	0.0584285	−0.35648	−0.43324	−0.13919	0.18211
3.0	0.0270452	−0.63213	−0.63413	−0.28862	0.07957
3.5	0.0129495	−0.85887	−0.76624	−0.42813	−0.01071
4.0	0.00646413	−1.03509	−0.84957	−0.54997	−0.08447
4.5	0.00336926	−1.16889	−0.90166	−0.65304	−0.14306
5.0	0.00183100	−1.26983	−0.93445	−0.73901	−0.18926
5.5	0.00103458	−1.34619	−0.95541	−0.81042	−0.22582
6.0	0.000605831	−1.40437	−0.96904	−0.86978	−0.25497
6.5	0.000366481	−1.44911	−0.97808	−0.91932	−0.27844
7.0	0.000228316	−1.48388	−0.98420	−0.96087	−0.29754
7.5	0.000146082	−1.51118	−0.98842	−0.99594	−0.31325
8.0	0.0000957511	−1.53285	−0.99137	−1.02573	−0.32629
8.5	0.0000641522	−1.55021	−0.99348	−1.05119	−0.33722
9.0	0.0000438471	−1.56426	−0.99501	−1.07308	−0.34646
9.5	0.0000305191	−1.57573	−0.99614	−1.09202	−0.35434
10	0.0000215989	−1.58517	−0.99697	−1.10849	−0.36111
11	0.0000113111	−1.59954	−0.99808	−1.13556	−0.37205
12	0.00000623921	−1.60972	−0.99874	−1.15666	−0.38043
13	0.00000359810	−1.61710	−0.99915	−1.17340	−0.38698
14	0.00000215643	−1.62257	−0.99941	−1.18688	−0.39220
15	0.00000133655	−1.62670	−0.99958	−1.19789	−0.39642
16	0.000000853240	−1.62988	−0.99970	−1.20699	−0.39988
17	0.000000559150	−1.63237	−0.99978	−1.21459	−0.40275
18	0.000000375078	−1.63433	−0.99983	−1.22101	−0.40516
19	0.000000256927	−1.63591	−0.99987	−1.22647	−0.40721
20	0.000000179350	−1.63718	−0.99990	−1.23115	−0.40895
∞	0	−1.64474	−1.00000	−1.27551	−0.42517

TABLE III

Functions for Density Ratio in Expansion of Air, cf. Equations (38) and (40)

M	$(1+\frac{1}{2}\lambda M^2)^{-1/\lambda}$	r_1	r_2	r_3	r_4
0	1.000000	0	0	0	0
0.5	0.885120	−0.01002	−0.01617	−0.00357	−0.00422
1.0	0.633441	−0.11778	−0.17483	−0.04390	−0.04770
1.5	0.393772	−0.38574	−0.51357	−0.15311	−0.14865
2.0	0.228395	−0.74395	−0.88593	−0.35571	−0.27226
2.5	0.130003	−1.09587	−1.18124	−0.49551	−0.38255
3.0	0.0747528	−1.39155	−1.38080	−0.66583	−0.46659
3.5	0.0440414	−1.62159	−1.50642	−0.81472	−0.52604
4.0	0.0267357	−1.79451	−1.58359	−0.93979	−0.56686
4.5	0.0167419	−1.92299	−1.63100	−1.04295	−0.59471
5.0	0.0108029	−2.01849	−1.66050	−1.12753	−0.61383
5.5	0.00716857	−2.08997	−1.67919	−1.19691	−0.62710
6.0	0.00488057	−2.14399	−1.69127	−1.25406	−0.63645
6.5	0.00340131	−2.18529	−1.69924	−1.30142	−0.64315
7.0	0.00242106	−2.21724	−1.70462	−1.34093	−0.64802
7.5	0.00175663	−2.24223	−1.70831	−1.37413	−0.65161
8.9	0.00129685	−2.26200	−1.71090	−1.38559	−0.65430
8.5	0.000972612	−2.27780	−1.71274	−1.42617	−0.65634
9.0	0.000739965	−2.29056	−1.71407	−1.44671	−0.65791
9.5	0.000570372	−2.30096	−1.71504	−1.46445	−0.65913
10	0.000444937	−2.30950	−1.71577	−1.47985	−0.66009
11	0.000279566	−2.32250	−1.71674	−1.50510	−0.66146
12	0.000182334	−2.33167	−1.71731	−1.52475	−0.66235
13	0.000122782	−2.33832	−1.71766	−1.54030	−0.66296
14	0.0000849978	−2.34324	−1.71788	−1.55281	−0.66337
15	0.0000602785	−2.34696	−1.71803	−1.56302	−0.66367
16	0.0000436654	−2.34982	−1.71813	−1.57144	−0.66388
17	0.0000322315	−2.35205	−1.71820	−1.57848	−0.66404
18	0.0000241940	−2.35381	−1.71825	−1.58441	−0.66416
19	0.0000184361	−2.35522	−1.71828	−1.58946	−0.66425
20	0.0000142403	−2.35636	−1.71831	−1.59379	−0.66432
∞	0	−2.36313	−1.71839	−1.63471	−0.66463

TABLE IV

Functions for Area Ratio in Expansion of Air, cf. Equations (45) and (46)

M	$\frac{1}{M}\left(\frac{1+\frac{1}{2}\lambda M^2}{1+\frac{1}{2}\lambda}\right)^{\frac{1}{2}+(1/\lambda)}$	f_1	f_2	f_3	f_4
0	∞	0.14556	0.22930	0.05251	0.06055
0.5	2.11617	0.07305	0.11124	0.02684	0.02993
1.0	1.00000	0	0	0	0
1.5	1.85836	0.09579	0.11836	0.03951	0.03573
2.0	2.67372	0.33519	0.36603	0.14848	0.11815
2.5	4.19667	0.61643	0.60154	0.29233	0.20621
3.0	6.77883	0.87242	0.77415	0.43987	0.27894
3.5	10.93978	1.0839	0.88765	0.57453	0.33268
4.0	17.3890	1.24088	0.95925	0.69062	0.37056
4.5	27.0553	1.36218	1.00400	0.78803	0.39685
5.0	41.1196	1.45342	1.03218	0.86884	0.41511
5.5	61.0484	1.52228	1.05018	0.93569	0.42789
6.0	88.6282	1.57467	1.06189	0.99110	0.43696
6.5	126.0004	1.61490	1.06966	1.03724	0.44349
7.0	175.697	1.64614	1.07492	1.07588	0.44825
7.5	240.677	1.67065	1.07855	1.10845	0.45177
8.0	324.360	1.69009	1.08109	1.13608	0.45442
8.5	430.665	1.70566	1.08290	1.15967	0.45643
9.0	564.048	1.71826	1.08421	1.17995	0.45797
9.5	729.534	1.72853	1.08518	1.19747	0.45918
10	932.759	1.73699	1.08590	1.21271	0.46013
11	1478.244	1.74986	1.08685	1.23773	0.46149
12	2259.20	1.75897	1.08742	1.25722	0.46237
13	3346.47	1.76557	1.08777	1.27268	0.46297
14	4824.31	1.77047	1.08799	1.28512	0.46339
15	6791.55	1.77417	1.08814	1.29528	0.46368
16	9362.89	1.77701	1.08824	1.30367	0.46389
17	12670.1	1.77924	1.08831	1.31068	0.46405
18	16863.5	1.78108	1.08835	1.31660	0.46417
19	22112.7	1.78240	1.08839	1.32163	0.46426
20	28608.5	1.78354	1.08841	1.32594	0.46433
∞	∞	1.79029	1.08850	1.36681	0.46464

TABLE V

Functions for Pressure Ratio Across a Shock Wave in Air, cf. Equations (62) and (63)

M_1	$\frac{2(1+\lambda)M_1^2-\lambda}{(2+\lambda)}$	Π_1	Π_2	Π_3	Π_4
1.0	1.00000	0	0	0	0
1.5	2.45485	−0.13030	0	−0.03350	0.01038
2.0	4.48829	−0.24560	0	−0.02392	0.07421
2.5	7.11037	−0.33989	0	0.00318	0.18519
3.0	10.31604	−0.41402	0	0.04179	0.35259
3.5	14.0937	−0.47118	0	0.08997	0.59120
4.0	18.4582	−0.51519	0	0.14688	0.91976
4.5	23.4047	−0.54930	0	0.21214	1.36032
5.0	28.9331	−0.57601	0	0.28555	1.93806
5.5	35.0435	−0.59717	0	0.36699	2.68119
6.0	41.7358	−0.61414	0	0.45639	3.62097
6.5	49.0100	−0.62791	0	0.55370	4.79163
7.0	56.8662	−0.63920	0	0.65891	6.23043
7.5	65.3044	−0.64856	0	0.77199	7.97757
8.0	74.3244	−0.65640	0	0.89292	10.07629
8.5	83.9264	−0.66301	0	1.02170	12.5728
9.0	94.1104	−0.66864	0	1.15832	15.5162
9.5	104.876	−0.67347	0	1.30278	18.9587
10	116.224	−0.67764	0	1.45507	22.9555
11	140.666	−0.68443	0	1.78312	32.8473
12	167.435	−0.68967	0	2.14247	45.6925
13	196.532	−0.69379	0	2.53310	62.0394
14	227.957	−0.69709	0	2.95501	82.4843
15	261.709	−0.69978	0	3.40819	107.6710
16	297.789	−0.70199	0	3.89264	138.291
17	336.197	−0.70383	0	4.40835	175.084
18	376.933	−0.70538	0	4.95533	218.837
19	419.997	−0.70669	0	5.53357	270.384
20	465.388	−0.70782	0	6.14307	330.608

TABLE VI

Functions for Density Ratio Across Shock Wave in Air, cf. Equations (64) and (65)

M_1	$\frac{(2+\lambda)M_1^2}{\lambda M_1^2+2}$	D_1	D_2	D_3	D_4
1.0	1.00000	0	0	0	0
1.5	1.86745	0.43246	0.86745	0.19525	0.25948
2.0	2.68161	0.67716	1.68161	0.42186	0.63396
2.5	3.35955	0.76183	2.35955	0.67319	1.17296
3.0	3.89436	0.75227	2.89436	0.94903	1.94519
3.5	4.30785	0.69872	3.30785	1.21585	3.03647
4.0	4.62669	0.62993	3.62669	1.58451	4.54807
4.5	4.87402	0.56009	3.87402	1.94948	6.59610
5.0	5.06780	0.49530	4.06780	2.34866	9.31140
5.5	5.22139	0.43767	4.22139	2.78348	12.8396
6.0	5.34459	0.38743	4.34459	3.25496	17.3413
6.5	5.44456	0.34403	4.44456	3.76386	22.9919
7.0	5.52659	0.30666	4.52659	4.31074	29.9822
7.5	5.59460	0.27449	4.59460	4.89601	38.6025
8.0	5.65151	0.24672	4.65151	5.51998	48.8191
8.5	5.69956	0.22268	4.69956	6.18289	61.1224
9.0	5.74046	0.20180	4.74046	6.88492	75.6788
9.5	5.77554	0.18357	4.77554	7.62620	92.7545
10	5.80583	0.16760	4.80583	8.40685	112.6310
11	5.85516	0.14112	4.85516	10.08657	161.988
12	5.89324	0.12029	4.89324	11.9246	226.306
13	5.92322	0.10365	4.92322	13.9212	308.382
14	5.94723	0.09017	4.94723	16.0767	411.260
15	5.96674	0.07911	4.96674	18.3912	538.224
16	5.98280	0.06995	4.98280	20.8648	692.803
17	5.99618	0.06227	4.99618	23.4976	878.770
18	6.00744	0.05577	5.00744	26.2897	1100.14
19	6.01700	0.05023	5.01700	29.2410	1361.17
20	6.02519	0.04547	5.02519	32.3517	1666.36

TABLE VII

Functions for Temperature Ratio Across a Shock Wave in Air, cf. Equations (66) and (67)

M_1	$\frac{[2(1+\lambda)M_1^2-\lambda](2+\lambda M_1^2)}{(2+\lambda)^2M_1^2}$	τ_2	τ_3	τ_3	τ_4
1.0	1.00000	0	0	0	0
1.5	1.31454	−0.14214	0	−0.22875	−0.24910
2.0	1.67373	−0.32058	0	−0.44578	−0.55976
2.5	2.11646	−0.51438	0	−0.67002	−0.98777
3.0	2.64769	−0.69544	0	−0.90724	−1.59260
3.5	3.27162	−0.85316	0	−1.16188	−2.44527
4.0	3.98950	−0.98540	0	−1.43763	−3.62831
4.5	4.80193	−1.09437	0	−1.73734	−5.23577
5.0	5.70921	−1.18365	0	−2.06312	−7.37334
5.5	6.71153	−1.25687	0	−2.41649	−10.1584
6.0	7.80898	−1.31715	0	−2.79857	−13.7203
6.5	9.00165	−1.36709	0	−3.21016	−18.2003
7.0	10.28956	−1.40876	0	−3.65183	−23.7517
7.5	11.6728	−1.44377	0	−4.12402	−30.6249
8.0	13.1513	−1.47339	0	−4.62706	−38.7428
8.5	14.7251	−1.49863	0	−5.16119	−48.5496
9.0	16.3942	−1.52029	0	−5.72659	−60.1626
9.5	18.1587	−1.53898	0	−6.32342	−73.7958
10	20.0185	−1.55521	0	−6.95178	−89.6755
11	24.0242	−1.58182	0	−8.30345	−129.141
12	28.4113	−1.60253	0	−9.78211	−180.613
13	33.1799	−1.61564	0	−11.3881	−246.343
14	38.3299	−1.63210	0	−13.1217	−328.776
15	43.8613	−1.64286	0	−14.9830	−430.553
16	49.7742	−1.65174	0	−16.9722	−554.512
17	56.0686	−1.65915	0	−19.0893	−703.686
18	62.7444	−1.66540	0	−21.3343	−881.301
19	69.8017	−1.67072	0	−23.7074	−1090.78
20	77.2404	−1.67528	0	−26.2086	−1335.75

TABLE VIII

Functions for Mach Number After a Shock Wave in Air, cf. Equations (68) and (69)

M_1	$\left(\frac{\lambda M_1^2+2}{2(1+\lambda)M_1^2-\lambda}\right)^{\frac{1}{2}}$	m_1	m_2	m_3	m_4
1.0	1.000000	0	0	0	0
1.5	0.700575	0.44264	0.80996	0.12397	0.17864
2.0	0.576491	0.96828	2.25472	0.29220	0.56286
2.5	0.511511	1.34534	3.96352	0.52218	1.29906
3.0	0.473427	1.52916	5.63583	0.81562	2.55433
3.5	0.449186	1.56450	7.12487	1.17274	4.53773
4.0	0.432843	1.51075	8.38980	1.59250	7.49399
4.5	0.421326	1.41242	9.44102	2.07387	11.7057
5.0	0.412922	1.29693	10.3074	2.61601	17.4933
5.5	0.406600	1.17964	11.0208	3.21825	25.2154
6.0	0.401739	1.06807	11.6100	3.88011	35.2683
6.5	0.397915	0.96561	12.0994	4.60125	48.0863
7.0	0.394859	0.87330	12.5083	5.38140	64.1417
7.5	0.392379	0.79099	12.8525	6.22035	83.9445
8.0	0.390340	0.71800	13.1440	7.11797	108.042
8.5	0.388643	0.65339	13.3927	8.07413	137.021
9.0	0.387213	0.59622	13.6062	9.08875	171.504
9.5	0.386003	0.54558	13.7906	10.16177	212.153
10	0.384964	0.50064	13.9509	11.2931	259.667
11	0.383294	0.42504	14.2138	13.7307	378.274
12	0.382016	0.36458	14.4185	16.4013	533.674
13	0.381021	0.31569	14.5807	19.3046	732.822
14	0.380230	0.27572	14.7112	22.4407	983.273
15	0.379589	0.24269	14.8176	25.8093	1293.19
16	0.379065	0.21513	14.9056	29.4106	1671.35
17	0.378630	0.19192	14.9790	33.2444	2127.11
18	0.378266	0.17221	15.0410	37.3106	2670.46
19	0.377957	0.15534	15.0937	41.6094	3311.98
20	0.377692	0.14080	15.1389	46.1406	4062.84